

Section Modulus and Bending Inertia of Wings

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This document and more information can be found on the website Wingbike - a Human Powered Hydrofoil.

Abstract

When calculating the section modulus and bending inertia of a wing, calculations can be done in two ways. Firstly, representing the cross-section of the wing by a rectangular shape. Secondly, using numerical approximations (requiring Excel). In this paper, these two methods are presented:

Rectangular Shape (Based on Besnard & Brooks)

1. Section Modulus of a solid wing (eq 2)
2. Section Modulus of a hollow wing (eq 8)
3. Bending Inertia of a solid wing (eq 10)
4. Bending Inertia of a hollow wing (eq 14)

Numerical method

5. Section Modulus and Bending Inertia method 1 (solid & hollow wing)
6. Bending Inertia method 2 (solid & hollow wing)

The accuracy of the various methods is compared for a typical ClarkY foil.

For hollow foils, all equations are based on an uniform skin thickness. In appendix 1 of this paper, the effect of non-uniform skin distribution is discussed.

A quick reference guide of all the equations can be found in appendix 4.

Introduction

When calculating stress and deflection in wings, the section modulus and the bending inertia are required. For a beam with a rectangular cross-section the equations can be found in any Engineering textbook. However, the cross-section of a foil is usually asymmetrical and often hollow. This complicates the calculations.

By representing the foil cross-section by a rectangular, calculations will be greatly simplified and the outcome has a reasonable accuracy.

This paper aims at explaining the steps and assumptions that lead to these equations (sections 1, 2, 3 and 4).

When more accuracy is required and the coordinates (shape) of the cross-section are known, a fairly simple numerical scheme (in Excel) can be used to calculate the bending inertia and sectional modulus. Two methods will be presented, both valid for solid and hollow foils (section 5 and 6).

1. Section Modulus – solid

In any engineering handbook one can find the section modulus for a solid beam with a rectangular cross-section. We will use Wikipedia as it is the most universal source [1]:

$$S = \frac{BH^2}{6} \quad (1)$$

Where

S = section modulus	[m ³]
B = width	[m]
H = height	[m]

Since we will be working with the cross-section of foils, it is more appropriate to use other symbols: width is chord (C) and height is thickness (T).

Besnard [2] has shown that a cross-section of a foil can be represented by a rectangular, if the chord and

thickness are corrected in the following way: multiply T with 0,85 and multiply C with 0,75

Now equation (1) becomes:

$$S = \frac{0,75C(0,85T)^2}{6} = \frac{(0,75)(0,85)^2}{6} CT^2$$

$$= \frac{0,542}{6} CT^2 = 0,0903CT^2 \quad (2)$$

Brooks [3] has used another correction for the HQ family of foils by merely multiplying C with a factor 0,45

$$S = \frac{0,45CT^2}{6} = 0,075CT^2 \quad (3)$$

It is easy to see that the section modulus of a solid foil is approximately 45% to 54% (depending on the corrections) of the section modulus of a solid beam with the same dimensions, leading to approximately twice as much stress.

In the remainder of this document we will be mainly using the correction presented by Besnard [2].

The above equations are only valid for solid foils. This is mostly not the case.

2. Section Modulus – hollow

Going back to our engineering handbook (or Wikipedia) the section modulus of a hollow beam is:

$$S = \frac{BH^3 - bh^3}{6H} \quad (4)$$

Where

B = outer width	[m]
H = outer height	[m]
b = inner width	[m]
h = inner height	[m]

Again, using different symbols and correcting the chord and thickness with the relevant factors (according to Besnard) the equation becomes:

$$S = \frac{(0,75C)(0,85T)^3 - (0,75c)(0,85t)^3}{6(0,85T)} \quad (5)$$

$$= \frac{(0,75)(0,85)^3}{6(0,85)} \left\{ \frac{CT^3 - ct^3}{T} \right\}$$

$$S = 0,0903 \left\{ \frac{CT^3 - ct^3}{T} \right\}$$

Often c and t are not known, but usually the thickness of the skin (Sk) is known. With this we can substitute:

$$c = C - 2Sk$$

$$t = T - 2Sk$$

Now the equation becomes:

$$S = 0,0903 \left\{ \frac{CT^3 - (C - 2Sk)(T - 2Sk)^3}{T} \right\} \quad (7)$$

Since the chord of a foil is usually several times larger than the skin, the factor (C-2Sk) can be replaced by C with only introducing a small error. This simplifies the equation to:

$$= 0,0903 \left\{ \frac{CT^3 - C(T - 2Sk)^3}{T} \right\}$$

$$= 0,0903C \left\{ \frac{T^3 - (T - 2Sk)^3}{T} \right\}$$

$$S = 0,0903C \left\{ T^2 - \frac{(T - 2Sk)^3}{T} \right\} \quad (8)$$

(For Brooks' approximation replace 0,0903 by 0,075 in equation 8).

NOTE: the equation is valid for a uniform skin thickness distribution (independent of C and T). For tapered foils, the skin thickness may decrease towards the tip as C and T decrease along the span of the wing (see appendix 1 for the effect of non-uniform skin distribution).

Example

Let's take some typical values of a human powered hydrofoil and compare the different methods:

Chord	150 mm
Thickness	20 mm
Skin	2 mm

The table below shows the section modulus using the different methods for solid and hollow foils.

	Standard	Besnard	Brooks
c	1	0,75	0,45
t	1	0,85	
factor	1/6	0,0903	0,075
Solid	10.000	5.419	4.500
Hollow	5.017	2.718	2.257
approx	-	2.644	2.196

The last line in the table (approx) is the outcome of equation 8 and shows the marginal error that is introduced by simplifying equation 7.

3. Bending Inertia – solid

Similar to the method described above, one can derive a simplified equation that calculates the bending inertia for a wing (required for tip deflection calculations). From any Engineering handbook or Wikipedia [4], the bending inertia for a rectangular beam is:

$$I = \frac{BH^3}{12} \quad (9)$$

Again, changing the symbols and correcting C and T yields:

$$I = \frac{(0,75C)(0,85T)^3}{12} = 0,0384CT^3 \quad (10)$$

and is valid for a solid foil.

(for Brooks' approximation, replace 0,0384 by 0,0375 in equation 10).

4. Bending Inertia – hollow

The bending inertia of a hollow wing can be calculated in a very simple manner. Consider the wing's outer dimensions and calculate the bending inertia as if it were a solid foil (eq 10). Then consider the foil's inner

dimensions and calculate the bending inertia as if it were a solid foil too. Subtraction of both values leads to the bending inertia of the hollow wing.

$$I = 0,0384CT^3 - 0,0384ct^3 = 0,0384\{CT^3 - ct^3\} \quad (12)$$

Substituting the skin:

$$c = C - 2Sk$$

$$t = T - 2Sk$$

Now the equation becomes:

$$I = 0,0384\{CT^3 - (C - 2Sk)(T - 2Sk)^3\} \quad (13)$$

As previously mentioned, the chord of a wing is usually several times larger than the skin, thus the factor (C - 2Sk) can be simplified to C. This simplifies the equation

$$I = 0,0384\{CT^3 - C(T - 2Sk)^3\}$$

$$I = 0,0384C\{T^3 - (T - 2Sk)^3\} \quad (14)$$

(for Brooks' approximation, replace 0,0384 by 0,0375 in equation 14).

NOTE: again, the equation is valid for a uniform skin thickness distribution Independent of C and T). For tapered foils, the skin thickness may decrease towards the tip as C and T decrease along the span of the wing (see appendix 1 for the effect of non-uniform skin distribution).

Example

The same typical values of a human powered hydrofoil were used. The table below shows the bending inertia using the different methods for solid and hollow foils.

	Standard	Besnard	Brooks
factor	1/12	0,0384	0,0375
Solid	100.000	46.059	45.000
Hollow	50.165	23.106	22.574
approx	-	22.477	21.960

The last line in the table (approx) is again the outcome of equation 14 and shows the error that is introduced by simplifying the equation 13.

5. Bending Inertia – numerical method 1

The basis for this method is described by Drela [5] and it consists of three steps.

First calculate the area of the cross-section, then calculate the neutral surface. Finally the bending inertia can be calculated. The paper uses three integrals (equations 1, 2, and 3). Without going into the details of the theory, an explanation of a numerical method is given here.

The basis is to make a table in Excel (see appendix 2). In this example, we will be using the coordinates of the ClarkY foil as it is a very basic foil with a nice flat bottom.

Column 1, 2 and 3 are the coordinates that specify the shape of the cross-section. Where x is the coordinate along the chord, and Zu and Zl represent the upper and lower curves.

The first step is to calculate the area of the cross-section. This is done by calculating the area beneath the upper curve and beneath the lower curve by means of the Trapezoidal Rule [6]. The total surface of the cross-section is obtained by the summation of column 4 subtracted by the summation of column 5.

Next step is to calculate the neutral surface (z) in column 6. Calculate for each interval:

$$\frac{1}{2} [Z_u^2 - Z_l^2] \Delta x$$

After summation of column 6, divide this total by the area previously calculated. Now the neutral surface (z) is obtained.

Finally the bending inertia can be calculated. Calculate for each interval (column 7)

$$\frac{1}{3} [(Z_u - z)^3 - (Z_l - z)^3] \Delta x$$

The sum of column 7 yields the bending inertia.

For a ClarkY foil with a chord of 150 mm and a thickness of 20 mm, this leads to a bending inertia of 53.821 mm⁴.

As stated previously, the bending inertia of a hollow foil is calculated by calculating the bending inertia for the outer shell and the inner shell as if they were solid. Then subtracting both. This will not be shown in this paper but the outcome is (skin is 2mm): 26.818 mm⁴. The bending inertia of the hollow wing is 27.003 mm⁴.

With the figures obtained in this scheme, the section modulus can be calculated as well: dividing the bending inertia by y (= 4.398 mm³).

6. Bending Inertia – numerical method 2

The last method is a simplification of the previous method, but is surprisingly accurate. It is based on equation 7 of Drela [5]. Although it's a simplification, it still requires Excel.

First h has to be calculated which is defined as the maximum camber. In appendix 3, a table with 4 columns is presented. The first three columns again represent the coordinates of the foil. The fourth column is calculated for each interval:

$$\frac{1}{2} [Z_u + Z_l]$$

Finally the Excel function MAX is used to determine the maximum value in column 4, in this case 10.

Now the bending inertia can be calculated:

$$I = K_1 c t (t^2 + h^2) \quad (15)$$

As suggested in [5], K₁=0,036, one can now calculate I. In this case 54.000 mm⁴.

For a hollow foil, the calculation has to be performed for the inner dimensions again and then subtracted. The end result of this is 27.089 mm⁴.

As seen in the previous method, the section modulus can be calculated from the bending inertia if the neutral surface (z) is known. This second numerical method does not rely on the calculation of z, however for very flat foils, the average camber is similar to z.

Scientifically z and the average camber are not related, but for approximation purposes this assumption can be valid. The average camber is the average of column 4.

Note that in this example, h_{\max} corresponds with $(1/2)t$. This is because the ClarkY foil has a flat bottom. For highly chambered foils this will not be the case. Therefore, for fairly 'flat' foils, one can skip the calculation in Excel and use $h=(1/2)t$. Now equation 15 becomes:

$$I = \frac{5}{4} K_1 c t^3 \quad (16)$$

Since K_1 equals 0,036 we can re-write the equation to

$$I = 0,045 c t^3 \quad (17)$$

Conclusion

Brooks and Besnard provide simple and quick equations to calculate the section modulus and bending inertia of foils (both solid and hollow). For preliminary designs they are generally considered sufficiently accurate.

When more accuracy is required two numerical methods are available of which the second one proves to be surprisingly simple and accurate.

For a foil with $C=150\text{mm}$, $T=20\text{mm}$ and $Sk=2\text{mm}$:

Section Modulus summary				
	Besnard	Brooks	Num 1	Num 2
Solid	5.419	4.500	4.398	4.195
Hollow	2.644	2.196	-	-

Bending Inertia summary				
	Besnard	Brooks	Num 1	Num 2
Solid	46.059	45.000	53.821	54.000
Hollow	22.477	21.960	27.003	27.089

References

- [1] http://en.wikipedia.org/wiki/Section_modulus
- [2] E. Besnard, A. Schmitz, K. Kaups, G. Tzong, H. Hefazi, H.H. Chen, O. Kural, and T. Cebeci, "Hydrofoil Design and Optimization for Fast Ships," *Proceedings of the 1998 ASME International Congress and Exhibition*, Anaheim, CA, November 1998.
- [3] Brooks, A.N., "The 20-knot Human Powered Water Craft", Human Power, Vol.6 , No. 1, Spring 1987.
- [4] Wikipedia: http://en.wikipedia.org/wiki/List_of_area_moments_of_inertia
- [5] Drela, M, "Area and bending inertia of Airfoil Sections". MIT OpenCourseWare, Unified Engineering Course Notes, MIT Department of Aeronautics and Astronautics, <http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-01-unified-engineering-i-ii-iii-iv-fall-2005-spring-2006/systems-labs-06/sp110b.pdf>
- [6] Wikipedia: http://en.wikipedia.org/wiki/Trapezoidal_rule

Appendix 1: Effect of non-uniform skin distribution

The equations in this paper assumed a constant (uniform) skin thickness along the span of a tapered foil. Meaning that the thickness of the skin is independent of the chord and foil thickness (the skin at the root of the wing has the same dimension as at the tip). In some cases, the skin is scaled with the dimension of the chord (gradually decreasing towards the tip). So what is the effect of non-uniform skin thickness on the section modulus and bending inertia of a wing?

Section Modulus – hollow-non uniform skin thickness

Take equation (8) of this paper (based on Besnard, but also applies to Brooks if the appropriate factor is used):

$$S = 0,0903C \left\{ T^2 - \frac{(T - 2Sk)^3}{T} \right\}$$

We will assume that the skin (Sk) depends on the thickness of the foil (T) and is a fraction of it: $Sk = \Omega T$ (where Ω is a value $0 < \Omega < 1$). Now, as the thickness of the foil decreases towards the tip, so will the skin.

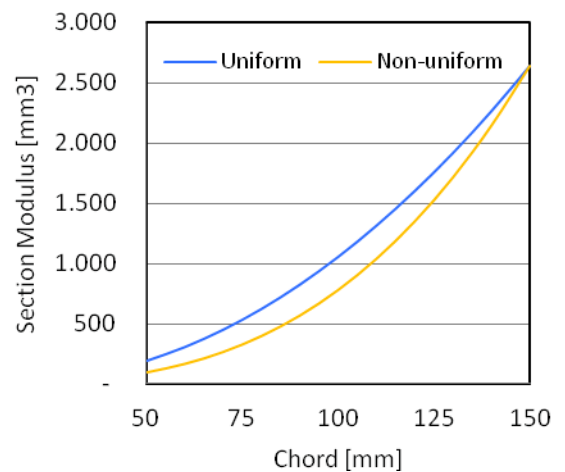
$$S = 0,0903C \left\{ T^2 - \frac{(T - 2\Omega T)^3}{T} \right\} = 0,0903C \left\{ T^2 - \frac{T^3(1 - 2\Omega)^3}{T} \right\} = 0,0903C \{ T^2 - T^2(1 - 2\Omega)^3 \}$$

Below is an example of uniform and non-uniform skin distribution for some typical values. The effect may seem limited, but at the tip, the difference is almost a factor 2!

Chord	150 mm
Thickness	20 mm
Skin	2 mm
Ω	0,1

Chord	Foil Thickn.	Uniform	Non-uniform
150	20,0	2.644	2.644
140	18,7	2.268	2.150
130	17,3	1.922	1.721
120	16,0	1.604	1.354
110	14,7	1.315	1.043
100	13,3	1.055	783
90	12,0	824	571
80	10,7	621	401
70	9,3	448	269
60	8,0	303	169
50	6,7	188	98

Uniform = constant thickness



Bending inertia – hollow-non uniform skin thickness

Take equation (14) of this paper (based on Besnard , but also applies to Brooks if the appropriate factor is used): :

$$I = 0,0384C\{T^3 - (T - 2Sk)^3\}$$

We will assume that the skin (Sk) depends on the thickness of the foil T) and is a fraction of it: $Sk = \Omega T$ (where Ω is a value $0 < \Omega < 1$).

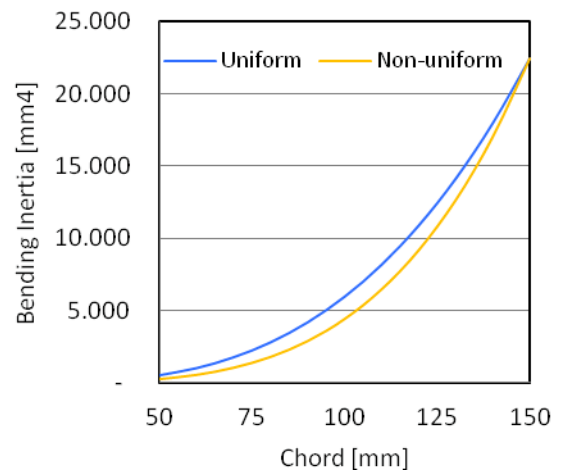
$$I = 0,0384C\{T^3 - (T - 2\Omega T)^3\} = 0,0384C\{T^3 - T^3(1 - 2\Omega)^3\}$$

Below is an example of uniform and non-uniform skin distribution for some typical values. The effect may seem limited, but at the tip, the difference is almost a factor 2!

Chord	150 mm
Thickness	20 mm
Skin	2 mm
Ω	0,1

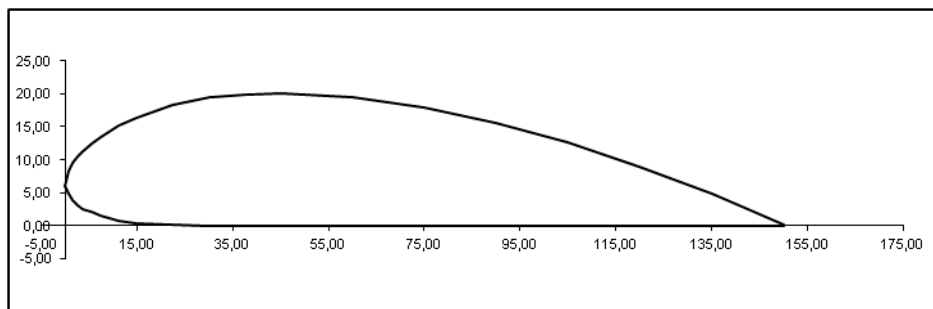
Chord	Foil Thicken.	Uniform	Non-uniform
150	20,0	22.487	22.487
140	18,7	18.006	17.064
130	17,3	14.164	12.686
120	16,0	10.912	9.211
110	14,7	8.200	6.503
100	13,3	5.980	4.442
90	12,0	4.202	2.914
80	10,7	2.818	1.819
70	9,3	1.778	1.066
60	8,0	1.032	576
50	6,7	532	278

Uniform = constant thickness



Appendix 2 Numerical calculation of bending inertia and section modulus – method 1

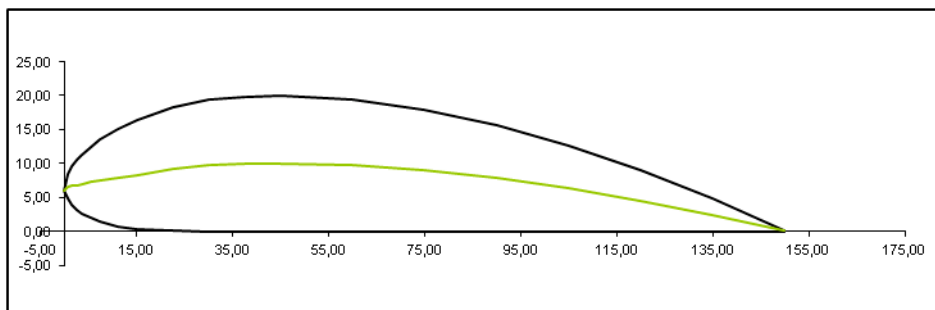
Wing Chord 150 mm
 Thickness adjust + 14,0 %
 Thickness = 20,00 mm



Coordinates			Trapezoidal		Neutr. S	Inertia
1	2	3	4	5	6	7
x [mm]	Zu [mm]	Zl [mm]	Area u [mm2]	Area l [mm2]	z [mm]	I [mm4]
0,00	5,98	5,98				
0,75	8,38	4,79	5,38	4,04	17,7	6,6
1,50	9,49	3,93	6,70	3,27	28,0	15,3
2,63	10,51	3,16	11,25	3,99	56,5	44,3
3,75	11,28	2,51	12,26	3,19	68,0	70,6
5,63	12,39	2,05	22,25	4,29	140,4	179,0
7,50	13,50	1,50	24,21	3,32	168,4	270,8
11,25	15,13	0,72	53,69	4,17	428,2	936,5
15,00	16,41	0,26	59,13	1,83	504,8	1337,0
22,50	18,26	0,05	130,00	1,15	1249,9	4035,3
30,00	19,42	0,00	141,28	0,19	1414,1	5128,6
37,50	19,83	0,00	147,18	0,00	1474,5	5561,5
45,00	20,00	0,00	149,36	0,00	1500,0	5750,8
60,00	19,49	0,00	296,15	0,00	2848,1	10397,3
75,00	17,98	0,00	281,03	0,00	2425,4	7676,5
90,00	15,64	0,00	252,18	0,00	1834,8	4783,8
105,00	12,56	0,00	211,54	0,00	1183,9	2892,3
120,00	8,92	0,00	161,15	0,00	597,2	2346,7
135,00	4,79	0,00	102,82	0,00	171,8	2207,0
150,00	0,21	0,00	37,44	0,00	0,3	180,6
			2105	29	16.112	53.821

area =	2.076 mm2	(sum Col. 4 - sum Col. 5)
z =	7,8 mm	(sum Col. 6 divided by area)
I =	53.821 mm4	(sum Col. 7)
S =	4.398 mm3	(I divided by y)

Appendix 3 Numerical calculation of bending inertia and section modulus – method 2



Coordinates			
1	2	3	4
x	Zu	Zl	h
[mm]	[mm]	[mm]	[mm]
0,00	5,98	5,98	5,98
0,75	8,38	4,79	6,58
1,50	9,49	3,93	6,71
2,63	10,51	3,16	6,84
3,75	11,28	2,51	6,90
5,63	12,39	2,05	7,22
7,50	13,50	1,50	7,50
11,25	15,13	0,72	7,92
15,00	16,41	0,26	8,33
22,50	18,26	0,05	9,15
30,00	19,42	0,00	9,71
37,50	19,83	0,00	9,91
45,00	20,00	0,00	10,00
60,00	19,49	0,00	9,74
75,00	17,98	0,00	8,99
90,00	15,64	0,00	7,82
105,00	12,56	0,00	6,28
120,00	8,92	0,00	4,46
135,00	4,79	0,00	2,39
150,00	0,21	0,00	0,10

MAX 10,00 = h

Bending inertia = 54.000 mm⁴

Average camber = z 7,13 mm

Section modulus = 4.195 mm³

!! Only for very flat foils

Appendix 4 Quick reference guide.

		Besnard	Brooks	Numerical 1	Numerical 2
Section modulus	Solid foil	$S = 0,0903CT^2$	$S = 0,075CT$	Calculate bending inertia numerically. Section modulus is bending inertia divided by y.	Section modulus is bending inertia divided by average camber (only for very flat foils!)
	Hollow foil (uniform skin)	$S = 0,0903C \left\{ T^2 - \frac{(T - 2Sk)^3}{T} \right\}$	$S = 0,075C \left\{ T^2 - \frac{(T - 2Sk)^3}{T} \right\}$	Beyond the scope of this paper.	N/A
	Hollow foil (non-uniform skin)	$S = 0,0903C \{ T^2 - T^2(1 - 2\Omega)^3 \}$	$S = 0,0705C \{ T^2 - T^2(1 - 2\Omega)^3 \}$	Beyond the scope of this paper.	N/A

		Besnard	Brooks	Numerical 1	Numerical 2
Bending inertia	Solid foil	$I = 0,0384CT^3$	$I = 0,0375CT^3$	Determine the area of the cross-section then determine the neutral surface. Bending inertia follows	Calculate maximum camber, then incorporate into equation 15 to calculate bending inertia.
	Hollow foil (uniform skin)	$I = 0,0384C \{ T^3 - (T - 2Sk)^3 \}$	$I = 0,0375C \{ T^3 - (T - 2Sk)^3 \}$	Calculate bending inertia of outer shell, then calculate bending inertia of inner shell. Subtraction gives bending inertia of hollow foil.	Calculate bending inertia of outer shell, then calculate bending inertia of inner shell. Subtraction gives bending inertia of hollow foil.
	Hollow foil (non-uniform skin)	$I = 0,0384C \{ T^3 - T^3(1 - 2\Omega)^3 \}$	$I = 0,0375C \{ T^3 - T^3(1 - 2\Omega)^3 \}$	Calculate bending inertia of outer shell, then calculate bending inertia of inner shell. Subtraction gives bending inertia of hollow foil.	Calculate bending inertia of outer shell, then calculate bending inertia of inner shell. Subtraction gives bending inertia of hollow foil.