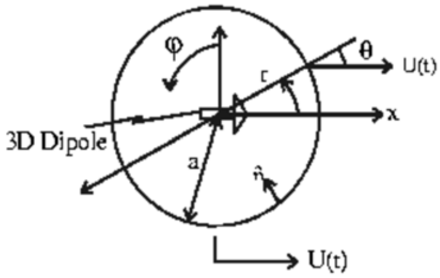
3.11 Unsteady Motion - Added Mass

D'Alembert: ideal, irrotational, unbounded, steady.

Example 1: Force on a sphere accelerating ($U = U(t)$, unsteady) in an unbounded fluid at rest. (at infinity)



K.B.C on sphere: $\frac{\partial \phi}{\partial r}\Big|_{r=a} = U(t)\cos\theta$

Solution: Simply a 3D dipole (no stream)

$$\phi = -U(t)\frac{a^3}{2r^2}\cos\theta$$

Check: $\frac{\partial \phi}{\partial r}\Big|_{r=a} = U(t)\cos\theta$

Hydrodynamic force:

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2\right) n_x dS$$

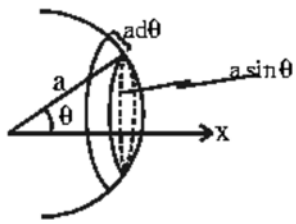
On r = a:

$$\frac{\partial \phi}{\partial r}\bigg|_{r=a} = -\dot{U}\frac{a^3}{2r^2}\cos\,\theta\big|_{r=a} = -\frac{1}{2}\dot{U}a\cos\theta$$

$$\nabla\phi|_{r=a} = \left(U\cos\theta, \frac{1}{2}U\sin\theta, 0\right) \underset{=\frac{\partial\phi}{\partial r}}{V_r} \underset{=\frac{1}{r}\frac{\partial\phi}{\partial\theta}}{V_\theta} \underset{=\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}}{V_\varphi}$$

$$\left|\nabla\phi\right|^2\big|_{r=a} = U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta; \hat{n} = -\hat{e}_r, n_x = -\cos\theta$$

$$\iint_B dS = \int_0^\pi (ad\theta)\,(2\pi a\sin\theta)$$



Finally,

$$F_x = (-\rho)\, 2\pi a^2 \int_0^{\pi} d\theta\,(\sin\theta)\left(\underbrace{-\cos\theta}_{n_x}\right)\left[\underbrace{-\frac{1}{2}\dot{U} a\cos\theta}_{\frac{\partial\phi}{\partial t}} + \frac{1}{2}\left(\underbrace{U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta}_{|\nabla\phi|^2}\right)\right]$$

$$= -\pi\rho\dot{U}a^3 \underbrace{\int_0^{\pi} d\theta \sin\theta\cos^2\theta}_{2/3} + \rho U^2\pi a^2 \underbrace{\int_0^{\pi} d\theta\sin\theta\cos\theta\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right)}_{=0 \text{ D'alembert revisited}}$$

$$F_x = -\dot{U}(t)\left[\underbrace{\rho\frac{2}{3}\pi a^3}_{\text{unit: mass}}\right] \qquad F_x = 0 \text{ if } \dot{U} = 0 \text{ steady (D'Alembert's Condition)}$$

General 6 degrees of freedom motions

Added mass matrix (tensor)

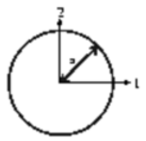
$$m_{ij}\ ; i, j = \underbrace{1,2,3,}_{\dot{u},\dot{v},\dot{w}}\ \underbrace{4,5,6}_{\dot{\Omega}_x,\dot{\Omega}_y,\dot{\Omega}_z}$$

m_{ij}: associated with force on body in i direction due to unit acceleration in j direction. For example, for a sphere:

$$m_{11} = m_{22} = m_{33} = {}^1\!/_2\ \rho\forall = (m_A) \ \text{ all other } m_{ij} = 0$$

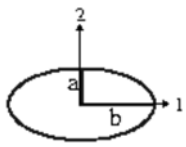
Some added masses of simple 2D geometries

- circle figure



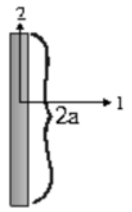
$$m_{11} = m_{22} = \rho\forall = \rho\pi a^2$$

- ellipse figure



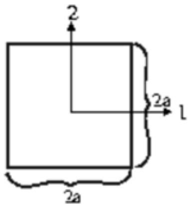
$$m_{11} = \rho\pi a^2, m_{22} = \rho\pi b^2$$

- plate figure



$$m_{11} = \rho\pi a^2, m_{22} = 0$$

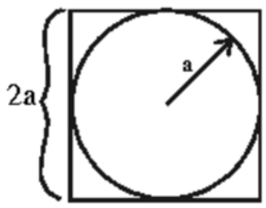
- square figure



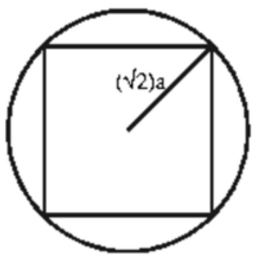
$$m_{11} = m_{22} \approx 4.754\rho a^2$$

A reasonable estimate for added mass od a 2D body is to use the displaced mass ($\rho\forall$) of an ``equivalent cylinder'' of the same lateral dimension or one that ``rounds off'' the body. For example, we consider a aquare:

1. inscribed circle: $m_A = \rho\pi a^2 = 3.14\rho a^2$.

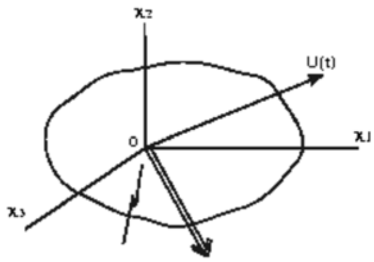


2. circumscribed circle: $m_A = \rho\rho\pi\left(\sqrt{2}a\right)^2 = 6.28\rho a^2$.



Arithmetic mean of 1) + 2) $\approx 4.71\rho a^2$.

General 6 degrees of freedom forces and moments on a rigid body moving in an unbounded fluid (at rest at infinity)



$$\vec{U}(t) = (U_1, U_2, U_3) \quad \text{Translation velocity}$$

$$\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6) \quad \text{Rotation (velocity) with respect to O}$$

Note: $OX_1X_2X_3$ fixed in the body.

Then (JNN §4.13)

- forces

$$F_j = -\underset{1.}{\dot{U}_i} m_{ji} - E_{jkl} \underset{2.}{U_i} \Omega_k m_{li} \text{ with } i = 1,2,3,4,5,6 \text{ and } j,k,l = 1,2,3$$

- moments

$$M_j = -\underset{3.}{\dot{U}_i} m_{j+3,i} - E_{jkl} \underset{\approx 2.}{U_i} \Omega_k m_{l+3,i} - E_{jkl} \underset{3.}{U_k} U_i m_{li} \text{ with } i = 1,2,3,4,5,6 \text{ and } j,k,l = 1,2,3$$

Einstein's Σ notation applies.

$$E_{jkl} = \text{"alternating tensor"} = \begin{cases} 0 & \text{if any } j,k,l \text{ are equal} \\ 1 & \text{if} j,k,l \text{ are in cyclic order, i.e., } (1,2,3), (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if} j,k,l \text{ are not in cyclic order i.e., } (1,3,2), (2,1,3), (3,2,1) \end{cases}$$

Note:

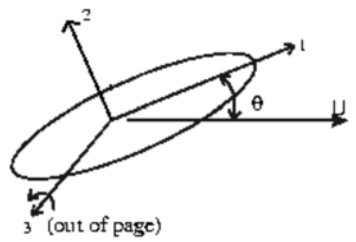
1.
   if $\Omega_k \equiv 0$, $F_j = -\dot{U}_i m_{ji}$ (as expected by definition of m_{ij}). Also if $\dot{U}_i \equiv 0$, then $F_j = 0$ for any U_i, no force in steady translation.

2.
   $B_l \sim U_i m_{li}$ ``added momentum'' due to rotation of axes, 2) $\sim \vec{\Omega} \times \vec{B}$ where $\vec{B}$ is linear momentum. (momentum from 1 coordinate into new x_j direction)

3.
   If $\Omega_k \equiv 0 : M_j = \underbrace{-\dot{U}_i m_{j+3,i} m_{ij}}_{\text{def.of}} - \underbrace{E_{jkl} U_k U_i m_{li}}_{\text{even with} \dot{U}=0, M_j \neq 0 \text{due to this term}}$.

   Moment on a body due to pure steady translation - ``Munk'' moment.

Example of Munk Moment - a 2D submarine in steady translation



$$U_1 = U\cos\theta$$
$$U_2 = -U\sin\theta$$

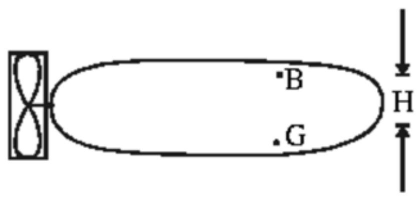
Consider steady motion: $\dot{U} = 0; \Omega_k = 0$. Then

$$M_3 = -E_{3kl}U_kU_im_{li}$$

For a 2D body, $m_{3i} = m_{i3} = 0$, also $U_3 = 0, i, k, l = 1, 2$. This implies that:

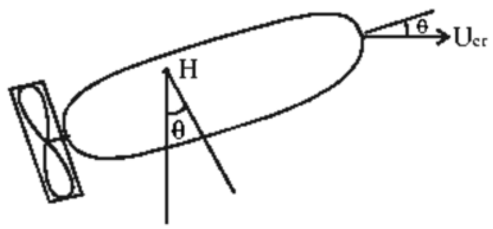
$$\begin{aligned} M_3 &= -\underbrace{E_{312}}_{=1}U_1\left(U_1m_{21} + U_2m_{22}\right) - \underbrace{E_{321}}_{=-1}U_2\left(U_1m_{11} + U_2m_{12}\right) \\ &= -U_1U_2\left(m_{22} - m_{11}\right) \\ &= U^2\sin\theta\cos\theta\left(\underbrace{m_{22} - m_{11}}_{>0}\right) \end{aligned}$$

Therefore, $M_3 > 0$ for $0 < \theta < \pi/2$ ("Bow up"). Therefore, a submarine under forward motion is unstable in pitch (yaw) (e.g., a small bow-up tends to grow with time), and control surfaces are needed:



- Restoring moment $\approx (\rho g \forall)$ Hsin θ.
- critical speed U_{cr} given by:

$$(\rho g\forall)\, H\sin\theta \geq U_{cr}^2\sin\theta\cos\theta\left(m_{22} - m_{11}\right)$$



Usually $m_{22} >> m_{11}, m_{22} \approx \rho\forall$. For small $\theta, \cos\theta \approx 1$. So, $U_{cr}^2 \leq gH$ or $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$. Otherwise, control fins are required.

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