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Lifting Line Theory of a Wing in Uniform Shear Flow*

By Kuninobu MORITA**

In the axial type fans and compressors, the velocity distribution relative to the blades is generally not uniform. Therefore, in the practical applications, it is important to find the characteristics of the wing in non-uniform flow.

In this investigation, the characteristics of a finite large aspect ratio wing in uniform shear flow with the velocity varying in the spanwise direction are analized by the singularity method which is different from the method adopted by von Kármán and Tsien. As the result, the characteristics of arbitrary wings are obtained concretely, and, at the same time, the physical difference between the characteristics of wings in uniform shear flow and those in uniform potential flow is clarified. Futhermore, the wing planform and its aerodynamic forces satisfying the condition of the minimum induced drag are given analytically.

1. Introduction

In general, the relative fluid flow toward the blades of axial-type hydrodynamical machines is not uniform in the spanwise direction, nor is it in the direction perpendicular to the blade surface. For example, even in the case of a helicopter-rotor, which may be called one of the most simple hydrodynamical machines, things do not change at all. The characteristics of a wing which is placed in such non-uniform flow are different from these of a wing in uniform flow. Therefore, in order to establish a precise design theory for hydrodynamical machines, it may be very important to analize the problem of this kind, i.e., the characteristics of a wing in non-uniform flow.

Lately, many theoretical investigations have been made for the blade characteristics of the axial-type hydrodynamical machines taking the flow non-uniformity into account. Honda[1] has analized, on the assumption of small perturbation, the characteristics of a wing between parallel walls, toward which the fluid flows with a spanwise-nonuniform velocity distribution. Namba and Asanuma[2][3] also have treated this kind of problems taking account of the fluid compressibility. Tsien[4], Sowyrda[5] and Jones[6] have theoretically studied a two-dimensional case in which the relative fluid velocity changes perpendicular to the airfoil surface.

These investigations are all concerned with axial-flow machines which are housed in the casings of cylindrical type. Contrary to these cylindrical casing type machines, in the case of airplane wings, helicopter-rotors, non-pressure type axial flow fans such as electric fans and so forth, one (outer) tip or both (outer and inner) tips of the blades are completely exposed to the surrounding flow field. Accordingly, in the theoretical analyses for such exposed-type axial fans, some different mathematical procedure must be adopted. In this connection, a few investigations have been made on the problem of a finite wing in inviscid shear flow.

The present author[8] has analized the characteristics of a finite low-aspect-ratio wing in uniform shear flow on the basis of the assumption which is analogous to the Jone's finite wing theory[7]. This theory is, however, applicable only to a low-aspect-ratio wing. Meanwhile, large-aspect-ratio wings or blades are in use in many axial-flow fans, blowers, pumps, and further, in airplanes and so forth. Therefore, we can not neglect the importance of obtaining the characteristics of a large-aspect-ratio wing in shear flow.

In the present paper, the characteristics of a finite but large-aspect-ratio wing in uniform shear flow with the velocity varying linearly in the spanwise direction are analyzed introducing the concept of the lifting line theory.

* Received 10th May, 1969.
** Research Fellow, Mechanical Engineering Research Laboratory, Hitach Ltd., Adachi-ku, Tokyo.

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Von Kármán and Tsien[9] developed the lifting line theory for a finite three-dimensional wing which is placed in shear flow with arbitrary velocity distribution. In their investigation, the wing characteristics are obtainable quantitatively, if one solves the integral equation and the differential equation with a given free flow velocity profile function. However, in their paper, the method of solving these equations is not presented, and hence, we can not know the quantitative wing characteristics immediately. Furthermore, their mathematical procedure is a Fourier integral method; the wing characteristics are analyzed supposing that the perturbation pressure may be represented in the Fourier integral. Accordingly, numerical works are rather complex.

In the present investigation, on the contrary, the singularity method is adopted: this method has close relation with the Prandtl's vortex method for the lifting line theory in uniform potential flow. As the result, the wing characteristics can be easily calculated by solving the system of algebraic equations, and the numerical examples for the rectangular wings and for the wings with minimum induced drag will be presented.

2. The equations of motion and the boundary conditions

In the present chapter, the equations of motion will be linearized on the assumption of small perturbation, and the Prandtl-type integral equation for a large-aspect-ratio wing will be derived.

Now, as shown in Fig. 1, let us take the Cartesian co-ordinate system (x, y, z) with the origin fixed in the mid-span of the wing. An oncoming free flow is parallel to the x-axis. Let station $y=-c$ denote the zero-velocity point of free flow. Then, the free flow velocity $U(y)$ will be given by the expression

$$U(y)=K(y+c) \qquad (1)$$

The present investigation is limited to the case that the zero-velocity point of free flow is not located on the wing surface ($c>d_0$). The analysis in the case of $c<d_0$ will be presented on another occasion.

The fluid is considered to be incompressible and inviscid. Let u, v and w denote the perturbation velocities in the x-, y- and z- directions, respectively, and assume that they are small compared with the free flow velocity $U(y)$. The pressure p represents the perturbation pressure. The equations of motion and the equation of continuity will, on the assumption of small perturbation, be written as follows.

$$U\frac{\partial u}{\partial x}+v\frac{dU}{dy}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \qquad (2)$$

$$U\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \qquad (3)$$

$$U\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \qquad (4)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (5)$$

where ρ denotes the density of the fluid.

The boundary condition is shown as follows: at infinity

$$x<0,\ (x^2+y^2+z^2)^{1/2}\to\infty;\ u, v, w, p\to 0 \qquad (6\cdot a)$$

$$x>0,\ (x^2+y^2)^{1/2}\to\infty;\ u, v, w, p\to 0 \qquad (6\cdot b)$$

In the lifting line theory, the precise boundary condition on the wing surface is not necessary, because the wing is represented by the "lifting-line" located at $-d_0\leqq y\leqq d_0$ on the y-axis, and the following assumption is introduced: an each spanwise wing section is assumed to have the same characteristics as that of a two-dimensional wing which is set at the "effective angle of attack". The "effective angle of attack" α_e is given by

$$\alpha_e=\alpha+\frac{w(0, y, 0)}{U(y)} \qquad (7)$$

where α denotes the geometrical angle of attack.

Eliminating u, w and p from Eqs. (2), (3), (4) and (5), we can get the following equation for v.

$$U(y)\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]-U''(y)v=0 \qquad (8)$$

where $U''(y)$ represents the second order derivative of $U(y)$. Because we take only uniform shear flow into consideration, $U''(y)$ is always equal to zero. Thus, Eq. (8) yields immediately

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0 \qquad (9)$$

From Eq. (3) and (4), p and w are given in the following forms in terms of v.

$$p=-\rho\int_{-\infty}^{y}U(\eta)\frac{\partial v}{\partial x}d\eta \qquad (10)$$

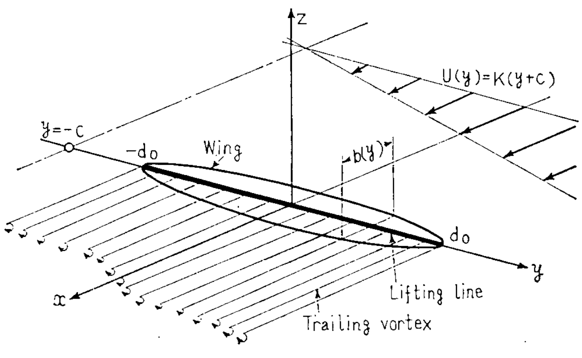

Fig. 1 A wing in uniform shear flow and system of coordinates

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$$w=\frac{1}{U(y)}\int_{-c}^{y}U(\eta)\frac{\partial v}{\partial z}d\eta \quad \cdots\cdots(11)$$

Now, because the governing differential equation for v is the Laplace's equation, v is supposed to be expressed in the similar form to one given in potential flow; as shown in Fig. 1, the trailing vortices form a sheet of vortex behind the wing, and hence, there must be a velocity discontinuity for v across the sheet. Accordingly, v may be given in the following form by the analogical inference from the lifting line theory in potential flow.

$$v(x,y,z)=\frac{z}{4\pi}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\frac{1}{(y-\eta)^2+z^2}\times\left(1+\frac{x}{\sqrt{x^2+(y-\eta)^2+z^2}}\right)d\eta \quad \cdots\cdots(12)$$

where $\Gamma(\eta)$ corresponds to the circulation of a wing in potential flow, and $d\Gamma/d\eta$ to the strength of trailing vortex. Of course, the lift distribution $l(y)$ can not be expressed in $\rho U(y)\Gamma(y)$, because free flow is not uniform. The relation between $l(y)$ and $\Gamma(y)$ will be established in the following way. By substituting Eq. (12) into Eq. (10), we obtain

$$p=-\frac{\rho z}{4\pi}\int_{-\infty}^{y}U(y)\left\{\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\times\frac{d\eta}{[x^2+(y-\eta)^2+z^2]^{3/2}}\right\}dy \quad \cdots\cdots(13)$$

The perturbation pressure p thus obtained has discontinuity between the upper ($z\to+0$) and the lower ($z\to-0$) "surface" of the lifting line. When we put $z\to\pm0$ in Eq. (13), p tends to infinity, whereas the integrated value of p remains finite. Now, let ε be a small quantity, and write p as $p_\pm$ when we put $z\to\pm0$. When we notice $p_+=-p_-$, we have

$$\frac{1}{2}l(y)=\int_{-\varepsilon}^{\varepsilon}p_-dx,\quad -\frac{1}{2}l(y)=\int_{-\varepsilon}^{\varepsilon}p_+dx \quad \cdots\cdots(14)$$

Furthermore, since we get

$$\lim_{z\to+0}\frac{z}{4\pi}\int_{-\varepsilon}^{\varepsilon}\left\{\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\cdot\frac{d\eta}{[x^2+(y-\eta)^2+z^2]^{3/2}}\right\}dx=\frac{1}{2}\frac{d\Gamma}{dy} \quad \cdots\cdots(15)$$

the local lift force $l(y)$ can be given

$$l(y)=\rho\int_{-d_0}^{y}U(\eta)\frac{d\Gamma}{d\eta}d\eta \quad \cdots\cdots(16)$$

when $U(\eta)$ is put to be constant U_0, that is, when the free flow is uniform, Eq. (16) agrees with Kutta-Joukowski theorem, $l(y)=\rho U_0\Gamma(y)$.

In the next place, "down wash" $w(0, y, 0)$ will be acquired. From Eq. (12), we can get $v(0, y, z)$ and $v_\infty(y, z)$ which is the perturbation velocity v far behind the lifting line, say

$$v(0,y,z)=\frac{z}{4\pi}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\frac{d\eta}{(y-\eta)^2+z^2} \quad \cdots\cdots(17)$$

$$v_\infty(y,z)=\frac{z}{2\pi}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\frac{d\eta}{(y-\eta)^2+z^2}=2v(0,y,z) \quad \cdots\cdots(18)$$

From these expressions, it can be concluded that the perturbation velocity v on the plane $x=0$ is one half of v_∞. This conclusion agrees perfectly with that by von Kármán and Tsien[9]. Similarly, substituting Eq. (12) into Eq. (11), we can get $w(x, y, z)$. According to the lifting line theory, however, only the "down wash" at the lifting line is needed. Von Kármán and Tsien[9] proved

$$w(0,y,z)=\frac{1}{2}w_\infty(y,z) \quad \cdots\cdots(19)$$

where $w_\infty(y,z)$ denotes w on the Trefftz plane. Since Eq. (11) holds even on the Trefftz plane, we have

$$w(0,y,z)=-\frac{1}{4\pi}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\frac{y-\eta}{(y-\eta)^2+z^2}d\eta+\frac{1}{8\pi(y+c)}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\log\left[\frac{(y-\eta)^2+z^2}{(c+\eta)^2+z^2}\right]d\eta \quad \cdots\cdots(20)$$

When we put $z\to0$ in Eq. (20), the down wash at the lifting line can be acquired, and the effective angle of attack is given.

$$\alpha_e=\alpha-\frac{1}{4\pi K(y+c)}\int_{-d_0}^{d_0}\frac{1}{y-\eta}\frac{d\Gamma}{d\eta}d\eta+\frac{1}{4\pi K(y+c)^2}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\log\left|\frac{y-\eta}{c+\eta}\right|d\eta \quad \cdots(21)$$

Now, the local lift force can be written in the following form, since in the lifting line theory, the characteristics of the wing at each section are supposed to be equal to those of a two-dimensional wing working at an angle of attack α_e.

$$l(y)=\frac{1}{2}\rho U^2(y)b(y)\kappa\alpha_e \quad \cdots\cdots(22)$$

where $b(y)$ and κ denote the local chord length and the lift curve slope $\partial C_L/\partial\alpha_e$ of the two-dimensional wing, respectively.

From Eqs. (16), (21) and (22), we have an integral equation whose unknown function is $\Gamma(y)$.

$$\int_{-d_0}^{y}U(\eta)\frac{d\Gamma}{d\eta}d\eta+\frac{U(y)b(y)\kappa}{8\pi}\left[\int_{-d_0}^{d_0}\frac{1}{y-\eta}\frac{d\Gamma}{d\eta}d\eta+\frac{1}{y+c}\int_{-d_0}^{d_0}\frac{d\Gamma}{d\eta}\log\left|\frac{y-\eta}{c+\eta}\right|d\eta\right]=\frac{\kappa}{2}U^2(y)b(y)\alpha(y) \quad \cdots\cdots(23)$$

Naturally, this equation agrees perfectly with the Prandtl's integral equation in potential flow, when the free flow velocity $U(y)$ tends to the constant value U_0.

In the next chapter, aerodynamic forces such as the lift force, the induced drag, etc. will be presented by using $\Gamma(y)$. Further, in the chapter 4. the integral equation will be solved.

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3. The aerodynamic forces

In the present chapter, the total lift force, the induced drag, etc. will be presented, and further, the condition for the induced drag to be a minimum will also be shown.

By integrating the lift distribution $l(y)$ given in Eq. (16) along the span, we get the total lift force.

$$L=\rho\int_{-d_0}^{d_0}\left\{\int_{-d_0}^{y}U(\eta)\frac{d\Gamma}{d\eta}d\eta\right\}dy \quad\cdots\cdots(24)$$

As shown in Fig. 2, the induced drag D_i can be calculated by integrating the x-component of $l(y)$ along the span.

$$D_i=-\int_{-d_0}^{d_0}l(y)\Delta\alpha_i(y)dy \quad\cdots\cdots(25)$$

where $\Delta\alpha_i$, which is an " induced angle of attack ", is given by

$$\Delta\alpha_i=\frac{w(0,y,0)}{U(y)} \quad\cdots\cdots(26)$$

The condition for the wing realizing the minimum induced drag is written as

$$\left.\begin{aligned}&L=\int_{-d_0}^{d_0}l(y)dy:\ \text{constant}\\&D_i=-\int_{-d_0}^{d_0}l(y)\Delta\alpha_i(y)dy:\text{minimum}\end{aligned}\right\} \quad\cdots\cdots(27)$$

This equation can be calculated by means of the variational principle.

$$\Delta\alpha_i(y)=\Delta\alpha_0\ \text{(constant)} \quad\cdots\cdots(28)$$

The minimum induced drag is given by the condition that the induced angle of attack $\Delta\alpha_i(y)$ is constant along the wing span. Accordingly, $\Gamma(y)$ which satisfies this condition is derived from Eq. (26),

$$-\frac{1}{4\pi}\int_{-d_0}^{d_0}\frac{1}{y-\eta}\frac{d\tilde{\Gamma}}{d\eta}d\eta+\frac{1}{4\pi(y+c)}\times\int_{-d_0}^{d_0}\frac{d\tilde{\Gamma}}{d\eta}\log\left|\frac{y-\eta}{c+\eta}\right|d\eta=U(y)\Delta\alpha_0 \quad\cdots\cdots(29)$$

where the symbol " $\sim$ " is affixed to $\Gamma(y)$ of the wing with the minimum induced drag. The induced angle of attack $\Delta\alpha_0$, which is an unknown constant in Eq. (29), can be determined such that $\tilde{\Gamma}(y)$ may satisfy the integral equation (23).

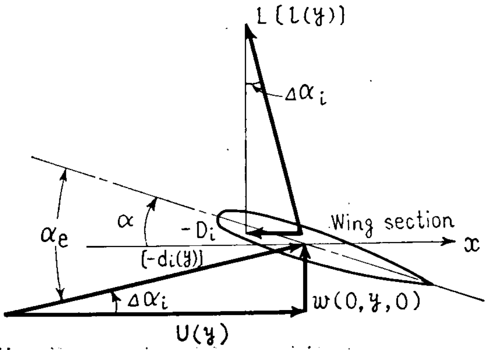

Fig. 2 The relation between the lift and the induced drag

In general, the lift distribution $l(y)$ must be always equal to zero at both tips. In potential theory, this requirement can be satisfied by making the circulation equal to zero at both tips. In uniform shear flow, however, this requirement is given by

$$\int_{-d_0}^{d_0}U(\eta)\frac{d\Gamma}{d\eta}d\eta=0 \quad\cdots\cdots(30)$$

since $l(-d_0)$ is always equal to zero. When the integral equation (23) is solved under the condition given in Eq. (30), we get the aerodynamic forces acting on the wing quantitatively. In the next chapter, the equation will be solved in the form of a Fourier series.

4. The solution of the integral equation and the aerodynamic forces

In the present chapter, we shall try to solve the integral equation by making a variable change. Furthermore, the integral equation for the optimum wing will be solved, and its characteristics will be calculated.

The form of the integral equation (23) resembles the Prandtl's integral equation in potential flow. Hence, $\Gamma(y)$ will be expressed in the form of a Fourier series, if the suitable variable change is made. Here, the following new variables ϕ and θ, and the dimensionless value λ are introduced.

$$y=d_0\cos\phi,\ \eta=d_0\cos\theta,\ \Gamma(\eta)=\Gamma(\theta),\ c=\lambda d_0 \quad\cdots\cdots(31)$$

When such variables are changed, the integral equations (23) and Eq. (30) become

$$\int_{\pi}^{\phi}(\lambda+\cos\theta)\frac{d\Gamma}{d\theta}d\theta+\mu_0 b_*(\phi)\left[\frac{\lambda+\cos\phi}{\pi}\times\int_0^{\pi}\frac{1}{\cos\theta-\cos\phi}\frac{d\Gamma}{d\theta}d\theta+\frac{1}{\pi}\int_0^{\pi}\frac{d\Gamma}{d\theta}\times\log\left|\frac{\cos\phi-\cos\theta}{\lambda+\cos\theta}\right|d\theta\right]=\frac{1}{2}Kd_0b_0\alpha_0\times(\lambda+\cos\phi)^2b_*(\phi)\alpha_*(\phi) \quad\cdots\cdots(32)$$

$$\int_0^{\pi}(\lambda+\cos\theta)\frac{d\Gamma}{d\theta}d\theta=0 \quad\cdots\cdots(33)$$

where

$$\left.\begin{aligned}&\mu_0=\frac{\kappa b_0}{8d_0}\\&b_*(\phi)=\frac{b(\phi)}{b_0}\\&\alpha_*(\phi)=\frac{\alpha(\phi)}{\alpha_0}\end{aligned}\right\} \quad\cdots\cdots(34)$$

b_0 : the chord length at $y=0$ $(\phi=\pi/2)$
α_0: the geometrical angle of attack at $y=0$ $(\phi=\pi/2)$ $\quad\cdots\cdots(35)$

The dimensionless value $\lambda=c/d_0$ is larger than

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unity, because the zero-velocity point of free flow is not on the surface of the wing.

We shall solve the above integral equation supposing that the solution $\Gamma(\theta)$ will be expressed in a Fourier series. Because $d\Gamma/d\theta$ is required to satisfy Eq. (33), we can write $d\Gamma/d\theta$ as

$$\frac{d\Gamma}{d\theta}=\frac{\kappa}{2}Kcb_0\alpha_0\frac{1}{\lambda+\cos\theta}\sum_{n=1}^{\infty}A_n\cos n\theta \qquad (36)$$

where the coefficients A_1, A_2, $\cdots$are all independent of each other. Substituting $d\Gamma/d\theta$ given in Eq. (36) into Eq. (32), we obtain the algebraic equality

$$\sum_{n=1}^{\infty}A_n\left[\frac{\sin n\phi}{2n}+\mu_0 b_*(\phi)G_n(\phi)\right]=\frac{(1+2a\cos\phi+a^2)^2}{2(1+a^2)^2}b_*(\phi)\alpha_*(\phi) \qquad (37)$$

where

$$G_n(\phi)=\frac{\sin n\phi}{2\sin\phi}+H_n(\phi)-Q_n+\frac{(-a)^{n+1}}{1-a^2} \qquad (38)$$

$$a=\lambda-\sqrt{\lambda^2-1}\quad(\lambda>1) \qquad (39)$$

and, $H_n(\phi)$ and Q_n are

$$H_n(\phi)=\frac{1}{2\pi}\int_0^{\pi}\frac{\cos n\theta}{\lambda+\cos\theta}\log|\cos\phi-\cos\theta|\,d\theta,\qquad Q_n=\frac{1}{2\pi}\int_0^{\pi}\frac{\cos n\theta}{\lambda+\cos\theta}\log(\lambda+\cos\theta)d\theta \qquad (40)$$

$H_n(\phi)$ and Q_n are calculated by the following reduction formulae,

$$\left.\begin{aligned}&H_{n+1}(\phi)+\frac{1+a^2}{a}H_n(\phi)+H_{n-1}(\phi)=-\frac{\cos n\phi}{n}\quad(n=1,2,3,\cdots)\\&H_0(\phi)=\frac{a}{1-a^2}[\log(1+2a\cos\phi+a^2)-\log 2]\\&H_1(\phi)=-\frac{1+a^2}{2(1-a^2)}\left[\log(1+2a\cos\phi+a^2)-\frac{2a^2}{1+a^2}\log 2\right]\end{aligned}\right\} \qquad (41)$$

$$\left.\begin{aligned}&Q_{n+1}+\frac{1+a^2}{a}Q_n+Q_{n-1}=-\frac{(-a)^n}{n}\quad(n=1,2,3,\cdots)\\&Q_0=\frac{a}{1-a^2}[2\log(1-a^2)-\log 2a]\\&Q_1=-\frac{1+a^2}{1-a^2}\left[\log(1-a^2)-\frac{a^2}{1+a^2}\log 2a\right]\end{aligned}\right\} \qquad (42)$$

When we want to obtain A_n from Eq. (37), only numerical calculation is available. The method of numerical procedure will be shown later.

The aerodynamic forces are presented in terms of A_n:

(i) The spanwise lift distribution

$$l(y)=\frac{1}{2}\rho\kappa K^2c^2b_0\alpha_0\sum_{n=1}^{\infty}\frac{A_n}{n}\sin n\phi\qquad\text{where, }\phi=\cos^{-1}\left(\frac{y}{d_0}\right) \qquad (43)$$

(ii) The total lift force

$$L=\frac{\pi}{4}\rho\kappa K^2c^2d_0b_0\alpha_0A_1 \qquad (44)$$

(iii) The induced drag

$$D_i=\frac{\pi}{32}\rho\kappa^2K^2c^2b_0^2\alpha_0^2\left(\frac{1+a^2}{1-a^2}\right)^2\left[\sum_{r=1}^{\infty}\frac{1}{r}\left\{\sum_{m=1}^{\infty}(-a)^{|m-r|}A_m\right\}^2+\left\{\sum_{m=1}^{\infty}(-a)^mA_m\right\}^2\log\left(\frac{1}{a^2}-1\right)\right] \qquad (45)$$

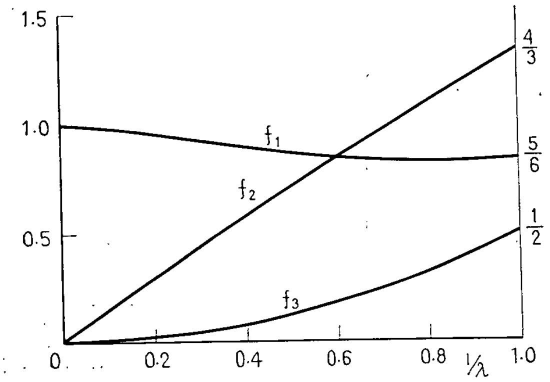

Fig. 3 Curves of f_1, f_2 and f_3

The induced drag, which is rather complex in form, becomes the same expression as that given by the potential theory, when we put $a\to 0$ ($1/\lambda\to 0$) with Kc kept constant U_0, say

$$D_i=\frac{\pi}{32}\rho\kappa^2U_0^2b_0^2\alpha_0^2\sum_{r=1}^{\infty}\frac{A_r^2}{r}$$

Dividing the total lift force and the induced drag by $1/2\rho K^2c^2S_W$ (S_W is wing area), we have the lift coefficient and the drag coefficient:

(iv) The lift coefficient

$$C_L=2\mu_0 R\!\!\!\!/\,\alpha_0A_1 \qquad (46)$$

(v) The induced drag coefficient

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$$C_{Di}=\pi\mu_0{}^2 A\!\!R\alpha_0{}^2\left(\frac{1+a^2}{1-a^2}\right)^2\left[\sum_{\gamma=1}^{\infty}\frac{1}{\gamma}\left\{\sum_{m=1}^{\infty}(-a)^{|m-\gamma|}A_m\right\}^2+\left\{\sum_{m=1}^{\infty}(-a)^mA_m\right\}^2\log\left(\frac{1}{a^2}-1\right)\right] \quad\cdots(47)$$

where $A\!\!R$ is the aspect ratio.

$$A\!\!R=\frac{4d_0{}^2}{S_W} \quad\cdots(48)$$

The integral equation (29) for the optimum wing becomes as follows, when the variables in Eq. (31) are changed.

$$-\frac{\lambda+\cos\phi}{2\pi}\int_0^{\pi}\frac{1}{\cos\theta-\cos\phi}\frac{d\tilde{\Gamma}}{d\theta}d\theta-\frac{1}{2\pi}\int_0^{\pi}\frac{d\tilde{\Gamma}}{d\theta}\log\left|\frac{\cos\phi-\cos\theta}{\lambda+\cos\theta}\right|d\theta=\varDelta\alpha_0Kd_0{}^2(\lambda+\cos\phi)^2 \quad\cdots(49)$$

$\tilde{\Gamma}(\theta)$ will also be given in the same form as the former one, because $\tilde{\Gamma}(\theta)$ should satisfy Eq. (33).

$$\frac{d\tilde{\Gamma}}{d\theta}=\frac{\kappa}{2}Kcb_0\alpha_0\frac{\lambda}{\lambda+\cos\phi}\sum_{n=1}^{\infty}\tilde{A}_n\cos n\theta \quad\cdots(50)$$

Substituting Eq. (50) into Eq. (49), we get

$$-\frac{\mu_0\alpha_0}{\varDelta\alpha_0}\sum_{n=1}^{\infty}\tilde{A}_nG_n(\phi)=\frac{(1+2a\cos\phi+a^2)^2}{2(1+a^2)^2} \quad\cdots(51)$$

where $G_n(\phi)$ is the same function as given in Eq. (38). The form of this equation is similar to that of the equation derived by the present author in his previous paper[8], and $\tilde{A}_n$ has been calculated analytically; $\tilde{A}_n$ of more than fourth order are all equal to zero, and A_1, A_2 and A_3 are

$$\tilde{A}_1=-\frac{\varDelta\alpha_0}{\mu_0\alpha_0}f_1,\quad \tilde{A}_2=-\frac{\varDelta\alpha_0}{\mu_0\alpha_0}f_2,\quad \tilde{A}_3=-\frac{\varDelta\alpha_0}{\mu_0\alpha_0}f_3,\quad \tilde{A}_4=\tilde{A}_5=\cdots=0 \quad\cdots(52)$$

where f_1, f_2 and f_3 are

$$\left.\begin{aligned}f_1&=\frac{(1-a^2)(1+a^2)^3-4a^4\log a}{(1-a^2)(1+a^2)^3-2a^2(1+a^2)^2\log a}\\ f_2&=\frac{3a(1-a^2)(1+a^2)-4a^3\log a}{(1-a^2)(1+a^2)^2-2a^2(1+a^2)\log a},\quad f_3=\frac{2a^2}{(1+a^2)^2}\end{aligned}\right\} \quad\cdots(53)$$

The curves of f_1, f_2, f_3 are shown in Fig. 3.

The induced angle of attack $\varDelta\alpha_0$ can be determined by the requirement that $\tilde{\Gamma}(\phi)$ should satisfy Eq. (37). Substituting Eq. (51) into Eq. (37), we can get

$$\frac{\varDelta\alpha_0}{\alpha_0}=-\frac{\mu_0(1+2a\cos\phi+a^2)^2b_*(\phi)\alpha_*(\phi)}{(1+a^2)^2\sum\limits_{n=1}^{3}\frac{f_n}{n}\sin n\phi+\mu_0(1+2a\cos\phi+a^2)^2b_*(\phi)} \quad\cdots(54)$$

Since $\varDelta\alpha_0$ is constant along the wing span, the right hand side of Eq. (54) should be constant for every value of ϕ. When we put $\phi=\pi/2(y=0)$ in Eq. (54), $\varDelta\alpha_0$ is

$$\frac{\varDelta\alpha_0}{\alpha_0}=-\frac{1}{1+\frac{1}{\mu_0}\left(f_1-\frac{1}{3}f_3\right)} \quad\cdots(55)$$

Accordingly, $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ are

$$\left.\begin{aligned}\tilde{A}_1&=\frac{1}{\mu_0}\frac{f_1}{1+\frac{1}{\mu_0}\left(f_1-\frac{1}{3}f_3\right)}\\ \tilde{A}_2&=\frac{1}{\mu_0}\frac{f_2}{1+\frac{1}{\mu_0}\left(f_1-\frac{1}{3}f_3\right)}\\ \tilde{A}_3&=\frac{1}{\mu_0}\frac{f_3}{1+\frac{1}{\mu_0}\left(f_1-\frac{1}{3}f_3\right)}\end{aligned}\right\} \quad\cdots(56)$$

The relation between the optimum wing planform $b_*(\phi)$ and its aerodynamic twist $\alpha_*(\phi)$ is given from Eqs. (54) and (55), as

$$b_*(\phi)=\frac{(1+a^2)^2}{(1+2a\cos\phi+a^2)^2}\frac{\frac{1}{\mu_0}\sum\limits_{n=1}^{3}\frac{f_n}{n}\sin n\phi}{\left[1+\frac{1}{\mu_0}\left(f_1-\frac{1}{3}f_3\right)\right]\alpha_*(\phi)-1} \quad\cdots(57)$$

Finally, the aerodynamic forces acting on the optimum wing are:

(i) The spanwise lift distribution

$$\left.\begin{aligned}\tilde{l}(y)&=\frac{1}{2}\rho K^2c^2b_0\alpha_0\frac{\kappa}{\mu_0+f_1-1/3f_3}\\ &\times\sum_{n=1}^{3}\frac{f_n}{n}\sin n\phi\\ \text{where}&\\ \phi&=\cos^{-1}\left(\frac{y}{d_0}\right)\end{aligned}\right\} \quad\cdots(58)$$

(ii) The total lift force

$$\tilde{L}=\frac{\pi}{4}\rho K^2c^2b_0d_0\alpha_0\frac{\kappa f_1}{\mu_0+f_1-(1/3)f_3} \quad\cdots(59)$$

(iii) The induced drag

$$\tilde{D}_i=\frac{\pi}{4}\rho K^2c^2b_0d_0\mu_0\alpha_0{}^2\frac{\kappa f_1}{\{\mu_0+f_1-(1/3)f_3\}^2} \quad\cdots(60)$$

In the present case, Eq. (60) can be easily obtained because D_i is given by the expression $-\varDelta\alpha_0L$. Of course, from Eq. (45) we can get the same expression as given in Eq. (60).

When we put $a\to0$ (or $1/\lambda\to0$) in these equations, all the results become coincident with those given by the lifting line theory in potential flow. For instance, the planform of the non-

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twisted wing and its lift distribution become both elliptic.

$$\left.\begin{array}{l} b_*(\phi)=\sin\phi \\ (a\to 0:\ Kc\to U_0,\ f_1\to 1,\ f_2, f_3\to 0) \\ \tilde{l}(\phi)=\dfrac{1}{2}\rho U_0{}^2 b_0\alpha_0\dfrac{\kappa}{1+\mu_0}\sin\phi \end{array}\right\} \quad \cdots\cdots\cdots(61)$$

5. Discussions about the wing characteristics

In the present chapter, the characteristics of the rectangular wings and of the wing with the minimum induced drag will be presented and discussed.

In the first place, the numerical method of calculating the Fourier coefficients [Eq. (37)] will be stated briefly. In the present investigation, the Glauert's method is adopted; the coefficients A_n are determined approximately by solving the following system of algebraic equations, supposing that A_n of more than Nth order are negligibly small.

$$\sum_{n=1}^{N} A_n\left[\frac{\sin n\phi_i}{2n}+\mu_0 b_*(\phi_i)G_n(\phi_i)\right] = \frac{(1+2a\cos\phi_i+a^2)^2}{2(1+a^2)^2}b_*(\phi_i)\alpha_*(\phi_i)$$

$$(i=1,2,3,\cdots,N) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(62)$$

where $\phi_i(i=1,2,\cdots,N)$ are N distinct values of $\phi(0\leqq\phi\leqq\pi)$, corresponding to N points along the span of the wing. The coefficients A_n are obtainable with fairly good accuracy when we put $N\cong 10$.

In Fig. 4 we illustrate the spanwise lift distribution patterns of the rectangular wing whose aspect ratio Æ is equal to 6.0. The parameter is the dimensionless velocity gradient $1/\lambda$ of the on-coming free stream. According as $1/\lambda$ tends to unity, i.e. the on-coming free flow velocity toward the starbord side of the wing becomes gradually larger, the local lift force acting on that side of the wing is getting larger and larger, whereas we have a symmetrical lift distribution in the case of $1/\lambda=0$ (potential flow).

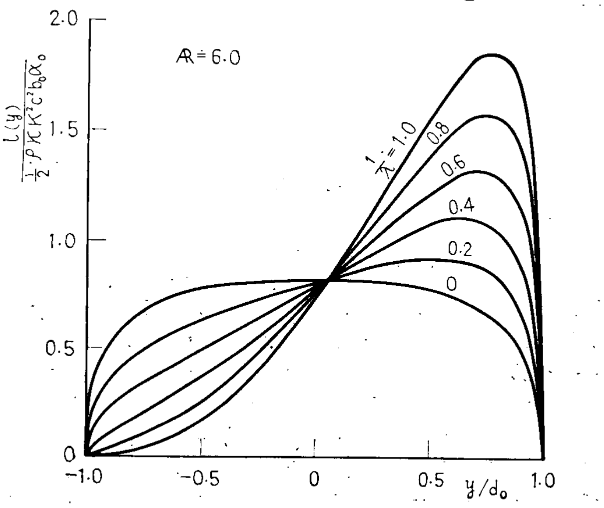

Fig. 4 The spanwise lift distributions of a rectangular wing (the constant aspect ratio)

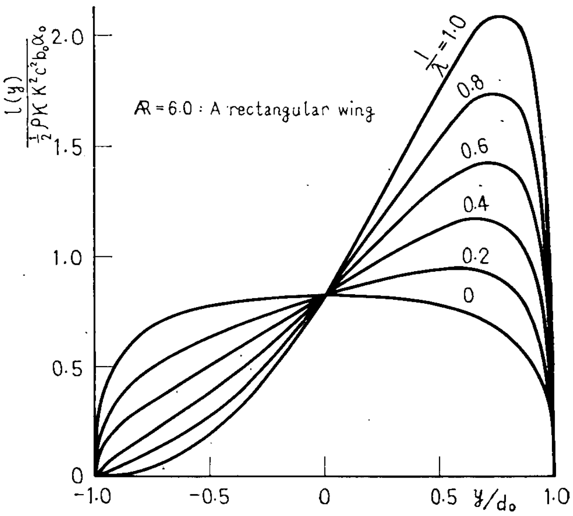

Fig. 5 The spanwise lift distributions of a rectangular wing obtained by the "quasi-potential theory"

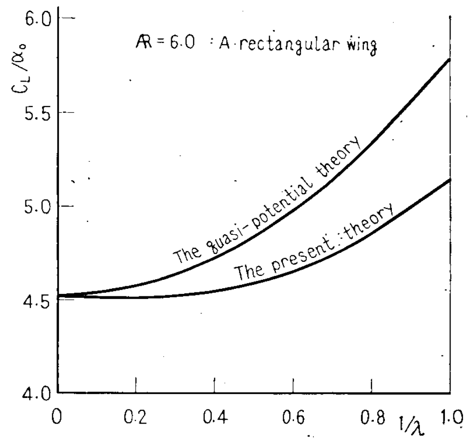

Fig. 6 The lift coefficients of a rectangular wing obtained by the present theory and the "quasi-potential theory"

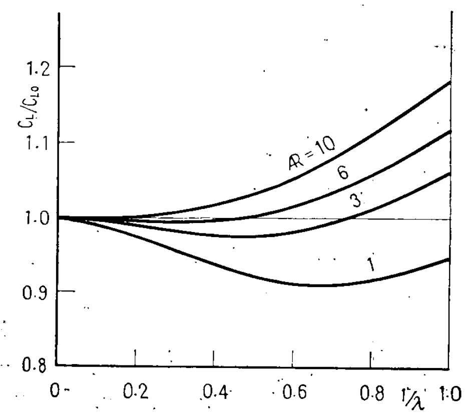

Fig. 7 The curves of C_L versus $1/\lambda$ (a rectangular wing)

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Now, we shall compare the wing characteristics presented in the present theory with those which are calculated approximately in the following "quasi-potential theory": we can easily calculate the local lift force of a wing in uniform flow whose velocity is U_0. Then, we assume that the local lift force acting at a station y in uniform shear flow is equal to that acting at the station y in uniform flow, if the local velocity $U(y)$ of the flow coming to the station y is equal to U_0. When such an approximation is made, we have

$$l(y)=\frac{1}{2}\rho\kappa K^2c^2b_0\alpha_0\left(1+\frac{\cos\phi}{\lambda}\right)^2\sum_{n=1}^{\infty}\times\frac{A_n^{(0)}}{n}\sin n\phi \qquad (63)$$

where $A_n^{(0)}$ are the coefficients of the Fourier series for the wing which is placed in uniform flow.

The lift distributions of the rectangular wing which are calculated by using Eq. (63) are shown in Fig. 5. The lift distributions given by such an approximate method are acceptable, when $1/\lambda$ is small. However, when $1/\lambda$ becomes large, the lift distributions given by the "quasi-potential theory" become large compared with those predicted by the present theory.

Figure 6 shows the relation between the lift coefficient C_L calculated by the present theory and that given by the quasi-potential theory.

According as $1/\lambda$ increases, C_L given by the present theory, at first, decreases and next increases, whereas the other increases monotonically.

The curve of C_L changes in accordance with the aspect ratio. In Fig. 7 we illustrate the lift coefficients of the rectangular wing. Here, C_{L0} denotes the lift coefficient of the wing which is placed in uniform flow. From this it follows that the decreasing rate of C_L becomes small according as $A\!\!\!R$ becomes large.

In Fig. 8 the patterns of the lift distribution are shown for several aspect-ratios. When the aspect ratio $A\!\!\!R$ increases, the maximum value of the local lift force $l(y)$ is getting larger and larger, and, at the same time, the location at which the maximum local lift force acts moves toward the tip.

In the next place, we shall show the lift distribution and the planform of the optimum wing, and discuss about them. Figure 9 shows the variation of the optimum lift distributions with $1/\lambda$. The case of $1/\lambda=0$ coincides with that of potential flow, and the lift distribution is elliptic. Naturally, when $1/\lambda$ increases, the lift distribution patterns become axisymmetric. The optimum wing planform also changes with $1/\lambda$.

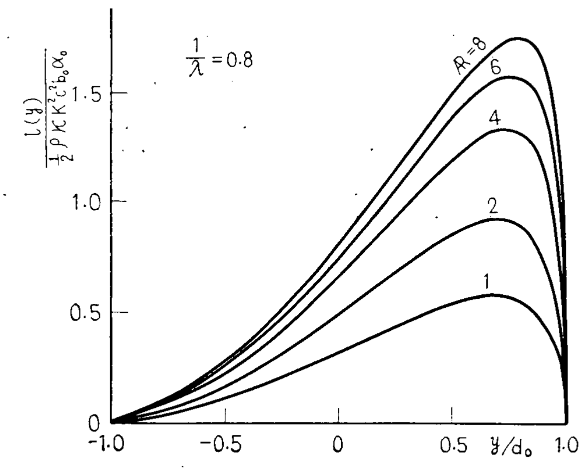

Fig. 8 The spanwise lift distributions of a rectangular wing (constant $1/\lambda$)

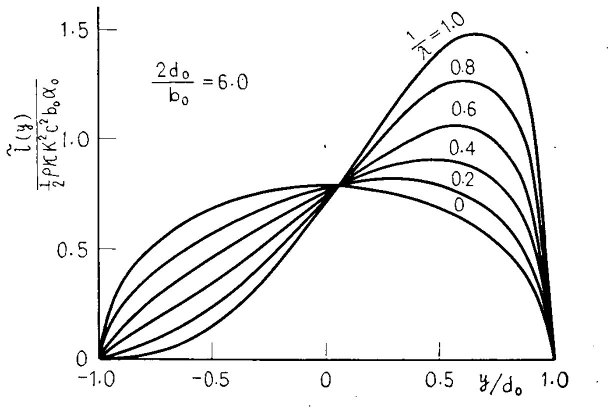

Fig. 9 The spanwise lift distributions of the optimum wing

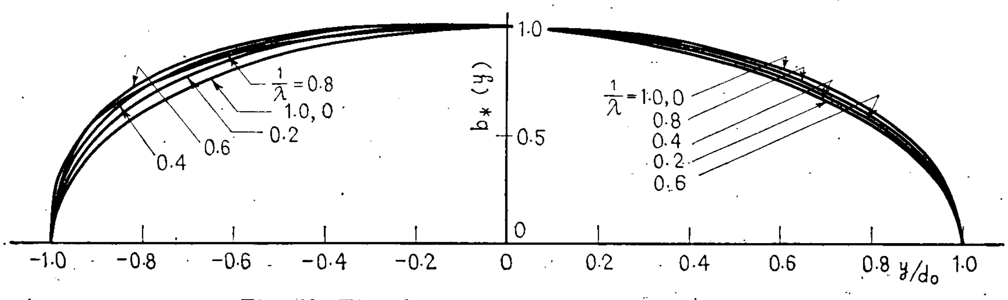

Fig. 10 The planforms of the optimum wing

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The planforms of the untwisted wing $[\alpha_*(\phi)=1]$ are illustrated in Fig. 10 for several values of $1/\lambda$. Here, only upper half of the wing planforms is shown. The wing planforms for $1/\lambda=0$ and for $1/\lambda=1$ are both elliptic: when the zero-velocity point of uniform shear flow comes from infinity to the wing tip, the wing planform gradually changes its shape from an ellipse, and when that point comes infinitely close to the wing tip, the planform again changes into an ellipse.

The above-mentioned are several numerical examples of the present lifting line theory for a wing in uniform shear flow.

6. Conclusions

The characteristics of a finite large aspect-ratio wing in uniform shear flow with the velocity varying in the spanwise direction have been analized by the singularity method which is different from that adopted by von Kármán and Tsien.

As the result, the spanwise lift distributions of the wing with an arbitrary planform can be expressed in the form of a Fourier series, whose coefficients are easily obtainable by solving a system of algebraic equations numerically. Accordingly, the aerodynamic forces such as the total lift L, the induced drag D_i, etc. are all calculated when we use those coefficients. The wing characteristics are influenced by the parameter $1/\lambda$, which represents the dimensionless velocity gradient of on-coming uniform shear flow. The characteristics of the rectangular wing are calculated and presented in Figs. 4, 6, 7, and 8.

In the present investigation, the characteristics of the wing with the minimum induced drag are also analized. In such a special case, the coefficients of the Fourier series can be given without any numerical procedure, that is, all the coefficients can be determined analytically. The planforms and the lift distributions of the optimum wing are influenced by the parameter $1/\lambda$, and are shown in Figs 9 and 10.

When the dimensionless free flow velocity gradient $1/\lambda$ tends to zero, results given by the present theory agree perfectly with those given by the Prandtl's lifting line theory in uniform potential flow. By the present theory, the physical meaning of difference between uniform shear flow and uniform potential flow is clarified.

Acknowledgement

The present author would like to acknowledge the kind instruction and interesting discussions given by Dr. Ei-ichi Hori and Dr. Shingo Ishizawa of Mechanical Engineering Research Laboratory, Hitachi Ltd.

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Discussions

Y. Sugiyama (Nagoya University):

(1) In the present investigation, $\Gamma(y)$ is acquired by calculating A_n from a system of algebraic equations (62) for given $b(y)$. When a new $b(y)$ is re-calculated from Eq. (23) by using $\Gamma(y)$ which is thus obtained, the new $b(y)$ is regarded to be a little different from the given one. Show us the re-calculated $b(y)$ for different free flow velocity gradients K [$b(y)$ may be smaller than the given one near the wing tips].

(2) In potential flow, the circulation Γ or the local lift force $l(y)$ should be zero at both wing tips. In uniform shear flow, we have

$$l(y)=\int_{-d_0}^{y}U(\eta)\frac{d\Gamma(\eta)}{d\eta}d\eta=[U(\eta)\Gamma(\eta)]^{y}_{-d_0}-\int_{-d_0}^{y}\Gamma(\eta)\frac{dU(\eta)}{d\eta}d\eta=U(y)\Gamma(y)-\int_{-d_0}^{y}\Gamma(\eta)\frac{dU(\eta)}{d\eta}d\eta,\quad [\Gamma(-d_0)=0]$$

from Eq. (16). When we notice $\Gamma(d_0)=0$, $l(d_0)$ is

$$l(d_0)=-K\int_{-d_0}^{d_0}\Gamma(\eta)d\eta \qquad [U=K(y+c)]$$

Since $l(d_0)=0$, the wing has the negative circulation. Is such a conclusion correct?

M. Namba (Kyushu University):

(3) Even when the free flow is a shear flow,

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the theorem of Kutta-Joukowski is expressed as

$$l(y)=\rho U(y)\Gamma_w(y)$$

where Γ_w denotes the wing circulation at each station. In the lifting line theory, this expression is assumed to be valid, and is applied instead of the boundary condition on the wing surface. Equation (22) is nothing but the Kutta-Joukowski theorem. Therefore, it is doubtless that the expression $l(y)=\rho U(y)\Gamma(y)$ is not valid, and, at the same time, it is not proper to insist as if Eq. (16) where the Kutta-Joukowski theorem.

The strength of the trailing vortices $d\Gamma/dy$ is the sum of $d\Gamma_w/dy$ and $\Gamma_w/U\cdot dU/dy$: the former denotes the spanwise increasing rate of the wing circulation, and the latter the additional vortices which are created in the wake due to the interference effect between vorticity dU/dy of free flow and the velocity discontinuity across the wing surface. Therefore, it seems to have little meaning that Γ, the integrated value of the trailing vortices, is introduced as a fundamental physical value. If the strength of the trailing vortices $d\Gamma/dy$ were introduced as a fundamental physical value, which is symbolized in terms of γ, for instance, the present investigation could be easily understood.

T. Nishiyama (Tohoku University):

(4) In the present investigation, the author often states, "when $1/\lambda$ tends to zero, the results agree perfectly with those given by the theory in potential flow". However, this statement does not verify that the results are general and proper, because in the lifting line theory, it is assumed that the local wing characteristics are similar to those of a two-dimensional wing worked at the induced angle af attack α_e.

(i) The present theoretical investigation has little significance, unless in shear flow the above assumption is proved reasonable for a large aspect-ratio wing.

(ii) In the lifting line theory, the three-dimensional characteristics of a wing are taken into consideration only in the effective angle of attack. Therefore, in regard to the lift curve slope $\partial C_L/\partial\alpha$, we should use not the value of a two-dimensional wing in potential flow, but that of a wing with infinite span in shear flow whose velocity varies linearly in the spanwise direction.

We can find little significance in the present theory, unless how to obtain this value is clarified.

(5) Although, in the present paper, we find the expression "by the analogical inference from the potential theory…", the author should not use such an expression without examining that such an inference is reasonable and correct even in shear flow.

(i) The author's expression that $\Gamma(\eta)$ in Eq. (12) is "corresponding to the circulation of the wing in potential flow" is ambiguous in physical sense, and hence, the following value which possesses a clear physical meaning should be introduced; from Eq. (3), we have

$$U(y)\frac{\partial}{\partial x}(v_+-v_-)+\frac{1}{\rho}\frac{\partial}{\partial y}(p_+-p_-)=0$$

Integrating this equation with respect to x

$$U(y)(v_+-v_-)=\frac{1}{\rho}\frac{\partial}{\partial y}\times\int(p_--p_+)dx=\frac{1}{\rho}\frac{\partial}{\partial y}L(y)$$

that is,

$$v_+-v_-=\frac{1}{\rho U(y)}\frac{\partial}{\partial y}L(y)$$

$L(y)$ represents the local lift force, and possesses a clear physical meaning.

$$\left[\frac{d\Gamma}{dy}\rightarrow\frac{1}{\rho U(y)}\frac{\partial}{\partial y}L(y)\right]$$

(ii) Equation (12) is obtainable by the following analytical procedure: by the Green's formula, we obtain

$$v(x,y,z)=\frac{1}{4\pi}\int_0^\infty d\xi\int_{-d_0}^{d_0}(v_+-v_-)\frac{\partial}{\partial n}\left(\frac{1}{R}\right)d\eta$$

$$R^2=(x-\xi)^2+(y-\eta)^2+2^2$$

Assuming that the wing is thin, we have

$$\frac{\partial}{\partial n}\doteqdot\frac{\partial}{\partial z}$$

Differentiating with respect to z and integrating with respect to ξ, v is written.

$$v(x,y,z)=\frac{z}{4\pi}\int_{-d_0}^{d_0}(v_+-v_-)\frac{1}{(y-\eta)^2+z^2}\times\left[1+\frac{x}{\sqrt{x^2+(y-\eta)^2+z^2}}\right]d\eta$$

(6) In potential flow, the equations

$$\partial p/\partial x=0,\qquad \partial u/\partial x=0$$

are both valid at far down stream. In shear flow, however, both equations can not be valid at the same time. This can be understood from Eq. (2). This is one of the eminent properties of shear flow. The investigations hitherto done are divided into the following two categories: one is made on the basis of the assumption that $\partial p/\partial x=0$ at far down stream, and the other on the assumption $\partial u/\partial x=0$. To which category does the present investigation belong?

(7) The questioner wants to know the physical meaning of the conclusion that the optimum wing planform becomes elliptic, when $1/\lambda$ is equal to unity, as well as when $1/\lambda$ is equal to zero.

(8) By the analitical method used in the present investigation, correctness of the assump-

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tion concerning the lift curve slope should be verified.

(9) As for the lift curve slope of a wing placed in uniform shear flow, that of the infinite wing which is placed in uniform shear flow should be applied. The author is required to show how to obtain it.

(10) Is the present linearized theory not contradictory to the fact that λ has no limitation $(1/\lambda=0\sim1)$? The author should show the physical reason why the elliptic planform is optimum when λ is equal to unity, taking it into consideration that the present theory is a linearized one.

(11) However, the questioner still deems it to be correct theoretically (not in point of accuracy) that concerning the lift curve slope, one should use that of an infinite wing which is placed in uniform shear flow. It is because the author treats only the free shear flow with rather small vorticity [K is small in Eq. (1)] that the lift curve slope in uniform shear flow is proved to be similar to that in potential flow. Therefore, the author is required to clarify the limit of K, considering the influence by the linearization of the equations of motion especially when λ is nearly equal to unity.

Author's closure

(1) The velocity gradient of free shear flow is characterized by the dimensionless parameter $1/\lambda$. Accordingly, curves of $b_*(y)$, which are calculated by the prescribed method, are presented in Append-Fig. 1 for several $1/\lambda$. We can find that these curves are nearly equal to unity along the span except extremely near the wing tip. When $1/\lambda$ is equal to unity, the accuracy deteriorates a little near the left tip. The author, however, thinks that this can be improved by adopting a larger N.

(2) In general, when the free flow is a uniform shear flow, $\Gamma(d_0)$ which is the sum of the trailing vortices is not equal to zero because of the interference effect between the trailing vorticies and the vorticies of free flow, say

$$K\int_{-d_0}^{d_0}\Gamma(\eta)d\eta=U(d_0)\Gamma(d_0)\neq0 \quad [\because l(d_0)=0]$$

Therefore, we can not draw such a conclusion as the questioner points out.

(3) Equation (16) simply expresses the relation between $\Gamma(y)$ and the spanwise lift distribution $l(y)$, and hence, does not correspond to the Kutta-Joukowski theorem. At the same time, the author should like to add that Eq. (22) is neither the Kutta-Joukowski theorem, because it implies that the local lift force $l(y)$ is simply proportional to the dynamic pressure $1/2\rho U^2(y)$, to the chord length $b(y)$ and to the lift coefficient $\kappa\alpha_e$ (as for the lift coefficient, we can use either the experimental value or the value given by the two-dimensional wing theory).

As the questioner has pointed out, this investigation should be understood more easily, if the strength of the trailing vortices were introduced as a fundamental physical value.

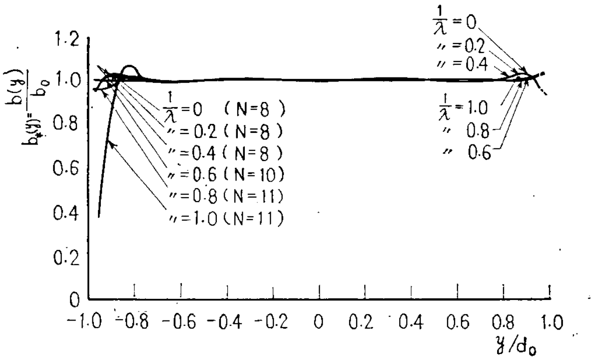
(4) The present author regards it as more convenient to answer the question (ii) first.

(ii) Exactly speaking, the author thinks that the lift curve slope of a wing in potential flow is not applicable to a wing in shear flow.

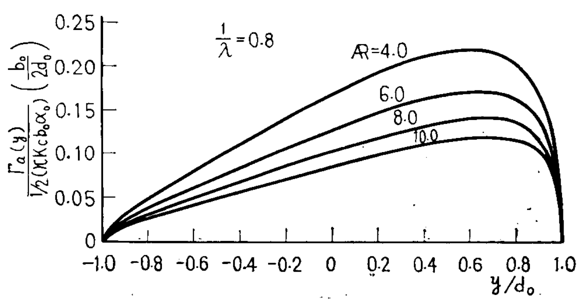
Honda[1] has analized the characteristics of the wing between parallel walls, toward which the fluid flows with a spanwise non-uniform velocity distribution, and has obtained the following expression in the case that the span is large compared with the chord length.

$$l(y)=\rho U^2(y)\pi\left\{\alpha+\frac{1}{2}\left(\frac{w_\infty}{U}\right)_{z=0}\right\}$$

From this equation, it is concluded that the lift curve slope $\partial C_L/\partial\alpha$ is equal to 2π, which is equivalent to that of a wing in potential flow. Although the Honda's theoretical model is different from the present one, no essential difference does exist between these two, because in the lifting line theory, three-dimensional properties of a wing are concentrated in the term w_∞.



Append.-Fig. 1 The re-calculated planforms of a rectangular wing



Append.-Fig. 2 The dimensionless circulations of a rectangular wing for several aspect ratios

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(i) We can write $l(y)$ given in Eq. (16) as

$$l(y)=\rho U(y)\Gamma_a(y)$$

where

$$\Gamma_a(y)=\frac{1}{U}\int_{-d_0}^{y}U(\eta)\frac{d\Gamma}{d\eta}d\eta$$

As Dr. Namba pointed out, Γ_a denotes the circulation of the wing element. In the lifting line theory, we adopt the Kutta-Joukowski theorem supposing that the spanwise increasing rate of $\Gamma_a(y)$ is small. Therefore, the spanwise increasing rate of Γ_a influences strongly upon the accuracy of the lifting line theory. The curves of $\frac{\Gamma_a}{(1/2)\kappa Kcb_0\alpha_0}\left(\frac{b_0}{2d_0}\right)$ which are calculated by the present theory are illustrated in Append.-Fig. 2 (the dimensionless circulation is multiplied by $b_0/2d_0$ in order to make easy comparison between the curves for wings with different aspect ratios). This figure shows that the increasing rate of Γ_a becomes smaller according as the aspect ratio becomes larger. Accordingly, for a large aspect ratio wing, the same kind of assumption as that adopted in potential flow is reasonable.

(5) (i) As for the fundamental physical value, it was better to introduce $\partial L/\partial y$ [or $l(y)$], or $d\Gamma/dy$ which Dr. Namba suggested, since Γ possesses little physical meaning.

(ii) Certainly, the author made a vague statement. Equation (12) was, however, obtained by the doublet distributed on the sheet of the trailing vortex. It should be explained clearly.

(6) If we want to develop a precise theory, it will be necessary to assume that either $\partial p/\partial x$ or $\partial u/\partial x$ is put to be zero at far down stream. However, since the present investigation is made assuming that the trailing vortex flows away to infinity parallel to the x-axis, p is given in Eq. (13). As the result, in the present theory $\partial p/\partial x=0$ is assumed at infinity.

(7) In a certain sense, the uniform shear flow is regarded to possess the nature of a potential flow, because the perturbation velocity v satisfies the Laplace's equation. This can be also clarified by the fact that the shape of the spanwise lift distribution is very similar to that given by the "quasi-potential theory": when we represent the lift distribution of a wing in uniform flow in $l_0(y)$, the lift distribution $l(y)$ of the wing in uniform shear flow is written approximately

$$l(y)\cong k_0U^2(y)l_0(y) \qquad k_0\text{: constant}\cdots\cdots\text{(i)}$$

On the other hand, form Eq. (22) $l(y)$ of the optimum wing is

$$l(y)=k_1U^2(y)b(y) \qquad k_1\text{: constant}\cdots\cdots\text{(ii)}$$

since α_e is constant along the wing span. Comparing Eq. (i) with Eq. (ii), the optimum wing planform is given as follows independently of $1/\lambda$.

$$b(y)\sim l_0(y)=k_2\sqrt{d^2-y^2} \qquad k_2\text{: constant}\cdots\cdots\text{(iii)}$$

However, as for the reason why the optimum wing planform agrees perfectly with an ellipse when λ is equal to unity, the author can not clarify it. Possibly, it may be caused by the properties of flow itself, or by the linearization of the Euler's equation (in the case of $\lambda\fallingdotseq 1$, the free flow velocity is very small near the left tip of the wing).

(8) In order to solve this problem, we try to analize the wing characteristics using the concept of the "lifting surface", and, in the next place, simplify the result assuming that the aspect ratio is large,

Now, we put

$$\Delta v(x,y,)=v_+-v_-,\quad \Delta p(x,y)=p_--p_+ \quad\cdots\cdots\text{(iv)}$$

where subscripts "+" and "−" are affixed to v at $z\to+0$ and at $z\to-0$, respectively. As prof. Nishiyama pointed out in the previous discussion, we have the following relation between Δv and Δp

$$\Delta v(x,y)=\frac{1}{\rho U(y)}\frac{\partial}{\partial y}\left[\int_{-b/2}^{x}\Delta p(x,y)dx\right] \quad\cdots\cdots\text{(v)}$$

Since the pressure p has discontinuity only across the wing surface, we obtain $\Delta v(x,y)$ in the region $x>b/2$ (or in the wake).

$$\Delta v(x,y)=\Delta v\left(\frac{b}{2},y\right)=\frac{1}{\rho U(y)}\frac{\partial}{\partial y}\times\left[\int_{-b/2}^{b/2}\Delta p(x,y)dx\right]=\frac{1}{\rho U(y)}\frac{dl}{dy} \quad\cdots\cdots\text{(vi)}$$

Because v satisfies the Laplace's equation, and further, it has discontinuity both across the wing and across the wake, we can obtain v by the doublet Δv distributed continuously in that region.

$$v(x,y,z)=\frac{z}{4\pi}\int_S\int\frac{\Delta v(\xi,\eta)}{R^3}d\xi d\eta\cdots\cdots\text{(vii)}$$

where,

$$R^2=(x-\xi)^2+(y-\eta)^2+z^2$$

Integration region S is the wing surface and the wake region. When we notice that $\Delta v(\xi,\eta)$ is a function of η only in the wake region ($\xi>b/2$, $-d_0\leqq\eta\leqq d_0$), we get w by substituting Eq. (vii) into Eq. (11).

$$w(x,y,z)=-\frac{1}{4\pi}\int_{-d_0}^{d_0}d\eta\int_{-b/2}^{b/2}\frac{\partial\Delta v}{\partial\xi}K(x,y,z;\xi,\eta)d\xi-\frac{1}{4\pi}\int_{-d_0}^{d_0}\Delta v\left(\frac{b}{2},\eta\right)\frac{y-\eta}{(y-\eta)^2+z^2}d\eta$$
$$+\frac{1}{8\pi(y+c)}\int_{-d_0}^{d_0}\Delta v\left(\frac{b}{2},\eta\right)\log\left|\frac{(y-\eta)^2+z^2}{(c+\eta)^2+z^2}\right|d\eta\cdots\cdots\text{(viii)}$$

where,

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$$\left.\begin{aligned}&K(x,y,z;\xi,\eta)=\frac{x-\xi}{(x-\xi)^2+z^2}\cdot\frac{y-\eta}{(y-\eta)^2+z^2}\left(R+\frac{z^2}{R}\right)-\frac{1}{y+c}\frac{x-\xi}{(x-\xi)^2+z^2}(R-R_c)\\&-\frac{x-\xi}{2(y+c)|x-\xi|}\log\left|\frac{R-|x-\xi|}{R+|x-\xi|}\frac{R_c+|x-\xi|}{R_c-|x-\xi|}\right|\\&R_c^2=(x-\xi)^2+(c+\eta)^2+z^2\end{aligned}\right\}\quad\text{(ix)}$$

Since $\Delta v(b/2,\eta)$ denotes the vorticity in the wake, the second and the third term of the left hand side of Eq. (viii) represent the induced velocity caused by the trailing vortex. If the wing surface is expressed by $z_a(x,y)$, the boundary condition on the wing surface is

$$w(x,y,0)=U(y)(\partial z_a/\partial x)\quad\text{(x)}$$

From Eqs (viii) and (x), we get the integral equation for the "lifting surface".

$$U(y)\frac{\partial z_a}{\partial x}=-\frac{1}{4\pi}\int_{-d_0}^{d_0}\int_{-b/2}^{b/2}\left(\frac{1}{\rho U(\eta)}\frac{\partial\Delta p}{\partial\eta}\right)K(x,y,0;\xi,\eta)d\xi d\eta$$
$$-\frac{1}{4\pi}\int_{-d_0}^{d_0}\left(\frac{1}{\rho U(\eta)}\frac{dl}{d\eta}\right)\frac{d\eta}{y-\eta}+\frac{1}{8\pi(y+c)}\int_{-d_0}^{d_0}\left(\frac{1}{\rho U(\eta)}\frac{dl}{d\eta}\right)\log\left|\frac{y-\eta}{c+\eta}\right|d\eta\quad\text{(xi)}$$

This is the fundamental equation for the lifting surface theory in uniform shear flow.

When the wing span $2d_0$ is very large compared with the chord length $b(y)$, the following approximation is possible.

$$(x-\xi)^2+(y-\eta)^2\cong(y-\eta)^2,\qquad(x-\xi)^2+(c+\eta)^2\cong(c+\eta)^2\quad\text{(xii)}$$

Then, Eq. (xi) is simplified as

$$\frac{1}{2\pi}\int_{-b/2}^{b/2}\frac{\Delta p(\xi,y)}{x-\xi}d\xi=\rho U^2(y)\left[-\frac{\partial z_a}{\partial x}+\frac{w_i(y)}{U(y)}\right]\quad\text{(xiii)}$$

where,

$$w_i(y)=-\frac{1}{4\pi}\int_{-d_0}^{d_0}\frac{1}{y-\eta}\left(\frac{1}{\rho U(\eta)}\frac{dl}{d\eta}\right)d\eta+\frac{1}{4\pi(y+c)}\int_{-d_0}^{d_0}\left(\frac{1}{\rho U(\eta)}\frac{dl}{d\eta}\right)\log\left|\frac{y-\eta}{c+\eta}\right|d\eta\quad\text{(xiv)}$$

Multiplying both sides of Eq. (xiii) by $\sqrt{(b/2+x)/(b/2-x)}$, and integrating it from $x=-b/2$ to $x=b/2$, Eq. (xiii) yields

$$l(y)=-\rho U^2(y)b\int_{-1}^{1}\sqrt{\frac{1+x_*}{1-x_*}}\frac{\partial z_a}{\partial x}dx_*+\pi\rho U^2(y)b\left(\frac{w_i(y)}{U(y)}\right)\quad\text{(xv)}$$

The first term of the right hand side of Eq. (xv) is equal to the lift force acting on a two-dimensional wing in potential flow. When a wing is thin and, at the same time, the geometrical angle of attack α is measured from zero-lift attitude, the first term yields $\pi\rho U^2(y)b\alpha(y)$. Therefore, Eq. (xv) becomes

$$l(y)=\pi\rho U^2(y)b\left[\alpha(y)+\frac{w_i(y)}{U(y)}\right]\quad\text{(xvi)}$$

From this equation, we can conclude that the assumption concerning the effective angle of attack is reasonable even in uniform shear flow.

(9) From Eq. (xvi) we can also conclude that the lift curve slope of the wing whose aspect ratio is large enough to neglect the higher order of AR^{-2} is equal to 2π, which coinncides with that of a two-dimensional wing in potential flow. Of course, because Eq. (xvi) is valid even for a wing with the infinite span, the lift curve slope of that wing is equal to 2π as well.

(10) Concerning this problem, the author answered in the previous paper [Bull. JSME, Vol. 13, No. 57 (1970), p. 362], and he will not explain it in detail here.

According as λ approaches unity, singularity $(v/U)_{y=-c}$ (v/U becomes infinite on the region $y=-c$) comes infinitely close to the wing tip. Therefore, accuracy of the results seems to deteriorate, and the optimum wing planform may not be elliptic when λ is equal to unity. However, the author thinks that the optimum wing planform is very similar to an ellipse, because this singularity gives little influence upon the whole flow field as mentioned in the previous paper.

(11) In general, it is not K, but the following non-dimensional value that characterizes the flow field essentially. Now, let the velocity Kc be the representative velocity and the semi-span d_0 be the representative length. Then, the free flow velocity $U(y)$ given in Eq. (1) can be written in the form,

$$U(y_*)=U_0\left(1+\frac{y_*}{\lambda}\right),\qquad\left(y_*=\frac{y}{d_0},\ U_0=Kc\right).$$

Accordingly, the value that governs the flow field is not the vorticity $K(=dU/dy)$ but the dimensionless vorticity $\dfrac{d}{dy_*}\left[\dfrac{U(y_*)}{U_0}\right]=\dfrac{1}{\lambda}$, and, at the same time, there is no limitation in vorticity

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K itself.

When we set up the equation for the " lifting line ", we made only the approximation that the aspect ratio was large, and got the conclusion that $\partial C_L/\partial \alpha$ was equal to 2π independently of λ. Furthermore, since correctness of linearization of the equation has been proved in the author's previous paper, there is no value on which we are required to impose restrictions.

With regard to the accuracy of the present lifting line theory, it would be difficult to give any quantitative answer, unless precise investigation is made. However, from the stand point of engineering, the author regards the present theory as applicable.

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