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Variational principle for determining the unknown wetted surface of a planing ship

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Abstract: A variational principle is presented to solve the problem of determining the unknown wetted surface of a planing ship on still water under gravity. The functional associated with the variational principle is expressed by the unknown wetted length distribution, the vortex line function, and the circulation distribution around the longitudinal sections, not by the bottom pressure distribution. In addition, the variational principle is adjoined with pseudoflow, here called reverse flow. The extremal of the functional satisfy the hull boundary integral equation in a similar way to that of lifting-surface theory and the elevated water surface condition along the spray root line. Both are sufficient to determine the wetted surface. A method of high aspect ratio approximation is also investigated. A simple relation is obtained between the wetted surface when moving and at standstill. The calculated shapes of the wetted surface of a planing plate agreed well with experimental results.

Key words: variational principle, unknown boundary problem, planing ship

Introduction

One of the remarkable distinctions of planing ships compared with conventional ones is that their wetted surface changes greatly between moving and stationary states. The more accurately we need to estimate the parameters of a planing ship, especially the moments, the more exactly we have to estimate the wetted surface. Therefore, we cannot substitute the wetted surface while moving as an estimate of the one at a standstill even within the framework of linearized theory. This means that the shape of the wetted surface and the usual flow parameters around a planing ship have to be determined simultaneously. This is called the unknown wetted surface problem of a planing ship.

Wagner[3] solved the unknown wetted surface problem for a wedge plunging to water, and applied this theory to the problem of a planing ship with a dead rise angle. The most difficult part of the problem is that the phenomenon of the piling up of the water must be considered. Matsumura and Kurotatsu[4] found another solution for the wetted surface of a low aspect ratio planing ship by supposing complicated self-similar flow. Matsumura and Mizutani[5] also determined the wetted surface of a high aspect ratio planing ship. They solved numerically a system of integro-differential equations expressing the unknown wetted length and the circulation distribution in a spanwise direction. Bessho[6] had previously discussed the wetted length perturbed from a standstill by solving an integral equation with the unknown bottom pressure under gravity.

Unless we assume some self-similarities in flow fields, we have to solve some integro-differential equations using special contrivances. This leads us to consider a variational principle so that the associated Euler's equations correspond to the integro-differential equations. If a suitable functional stating the variational principle is once discovered, the extremal of the functional can easily be obtained,[7,8] and that is the solution. We know of a few variational principles for similar but well-defined surface problems. That of Flax[9] led to a solution for an integral equation for a lifting surface

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under pressure. Bessho and Nomura[10] found a functional for the bottom height distribution of a planing ship so that the hull boundary condition was satisfied. In both of these variational principles, there appeared notions not only of true flow but also of pseudoflow, known as reverse flow, in order to force the conditions at infinity to be symmetric.

Using the two principles described above, we propose a variational principle to solve the unknown wetted surface problem of a planing ship in fixed conditions.

Integral equations

Consider a ship in planing conditions on a still water surface. We deal with a rectangular plate as a typical ship and force the plate to plane. We also consider a wetted plate set above the still water surface. The Cartesian coordinate system x, y, z is shown in Fig. 1. The still water surface is at $z = 0$ and the trailing edge of the plate is at $x = 0$. All of the quantities are made dimensionless by using b, the half-width of the plate, and U_∞, the velocity of the plate, unless otherwise specified. We symbolize the bottom height of the plate by $H(x, y)$. The wetted length distribution while moving is denoted by $l_w(y)$, while that at a standstill is denoted by $l_R(y)$. Alternatively, l_R is defined as the position where the plate cuts the still water surface.

In an irrotational flow field under gravity, the velocity potential Φ is represented as

$$\Phi = x + \phi \tag{1}$$

in a linearized manner. The perturbation potential ϕ connecting with the bottom pressure $p_0(x, y)$ becomes

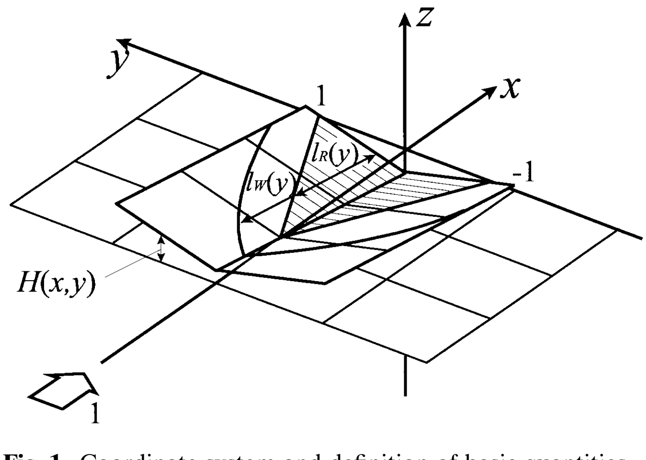

Fig. 1. Coordinate system and definition of basic quantities

$$\begin{aligned}\phi(x, y, z) &= \frac{1}{4\pi}\int_{-1}^{1}\int_{-l_w(\eta)}^{0} 2p_0(\xi, \eta) \\ &\times\left\{\frac{z}{(y-\eta)^2+z^2}\left(1+\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\right)\right. \\ &-\frac{K_0}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\oint_0^{\infty}\frac{e^{kz}\sec^3\theta}{k-K_0\sec^2\theta} \\ &\times\sin\big(k(x-\xi)\cos\theta\big)\cos\big(k(y-\eta)\sin\theta\big)dkd\theta \\ &-K_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{K_0 z\sec^2\theta}\sec^3\theta\cos\big(K_0(x-\xi)\sec\theta\big) \\ &\left.\times\cos\big(K_0(y-\eta)\sec^2\theta\sin\theta\big)d\theta\right\}d\xi d\eta\end{aligned} \tag{2}$$

so that it satisfies the linearized pressure and kinematic condition

$$[\mathrm{P}] \quad \frac{\partial\phi}{\partial x}+K_0 \quad h=0 \qquad \text{on } z=0 \tag{3}$$

$$[\mathrm{K}] \quad \frac{\partial h}{\partial x}-\frac{\partial\phi}{\partial z}=0 \qquad \text{on } z=0 \tag{4}$$

where K_0 stands for a dimensionless wave number defined as

$$K_0 \equiv \frac{gb}{U_\infty^2} \tag{5}$$

and h is wave height. The first term of eq. 2 corresponds to such a perturbation potential around a wing in unbounded flow. The second and third terms of eq. 2 express the local and free wave effects, respectively.

Imposing the hull boundary condition, we have an integral equation expressed by p_0 as in wing theory. However, we fail to construct any variational principle with an unknown boundary unless we resolve the unbounded behavior of $p_0\,(-l_w(x), y)$. In order to allow a variation of l_w for a functional, we get rid of the formulation expressed by p_0. Alternatively, it may be effective to express this by using the vortex line function $\mu(x, y)$ and the circulation distribution $\Gamma(y)$, as shown in Fig. 2. The former is defined as

$$\mu(x, y) \equiv \int_{-l_w(y)}^{x} 2p_0(\xi, y)d\xi \tag{6}$$

so that μ absorbs the singularity in pressure to the square-root, and the latter is defined as

$$\Gamma(y) \equiv \mu(0, y) \tag{7}$$

which is Kutta's condition. Differentiation of μ as a stream function yields the strength density vector $\boldsymbol{\gamma}$ of the vortex sheet:

$$\boldsymbol{\gamma} = -(\mathbf{k} \times \nabla)\mu \tag{8}$$

On the downstream surface, there is apparently a distribution of straight vortex lines of strength density $d\Gamma/dy$ flowing out from the transom stern.

Consequently, the integral equation connecting with the hull boundary condition becomes

$$\frac{\partial H}{\partial x}(x,y) = \fint_{-1}^{1} Q_x(x, y; 0, \eta)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{x\xi}(x, y; \xi, \eta)\mu(\xi, \eta)d\xi d\eta \tag{9}$$

where the kernel function is defined as

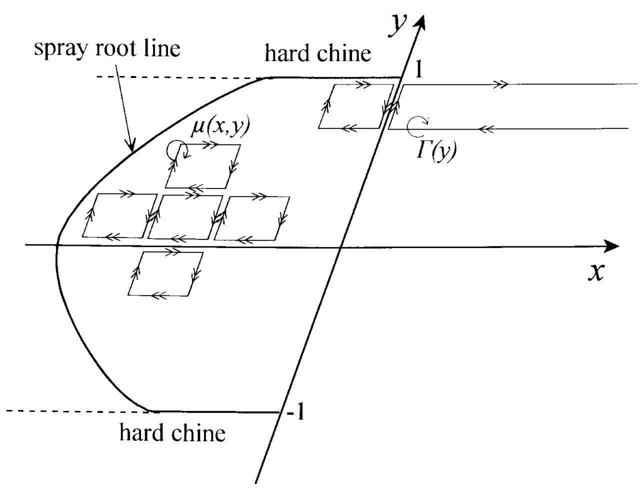

Fig. 2. Flow configuration around a planing ship represented by horseshoe vortices

$$\begin{aligned} Q&(x, y; \xi, \eta) \\ &\equiv \frac{1}{4\pi}\left\{ \frac{1}{(y-\eta)^2}\left(x-\xi+\sqrt{(x-\xi)^2+(y-\eta)^2} \right) \right. \\ &+ \frac{K_0}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \fint_0^{\infty} \frac{\sec^4\theta}{k-K_0\sec^2\theta} \\ &\times \cos\big(k(x-\xi)\cos\theta\big)\cos\big(k(y-\eta)\sin\theta\big)dk d\theta \\ &- K_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^4\theta \sin\big(K_0(x-\xi)\sec\theta\big) \\ &\left. \times \cos\big(K_0(y-\eta)\sec^2\theta\sin\theta\big)d\theta \right\} \end{aligned} \tag{10}$$

We note $l_w(y)$ at the lower bound of the integral.

In order to define the unknown wetted surface problem, we inevitably impose another condition, named the spray condition, at $x = l_w(y)$ as follows:

$$H(-l(y), y) = \fint_{-1}^{1} Q(-l_w(y), y; 0, \eta)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{\xi}(-l_w(y), y; \xi, \eta)\mu(\xi, \eta)d\xi d\eta \tag{11}$$

The right-hand side of eq. 11 represents the water elevation at the spray root line, which is obtained by integrating $\partial\phi/\partial z$, as is seen in eq. 4, from an upstream point to the spray root line.[11]

Two integral equations, eqs. 9 and 11, under Kutta's condition (eq. 7) are sufficient to solve the unknown wetted surface problem.

Variational principle

We now state a variational principle for the unknown boundary problem. For this, we seek a functional Π which yields eqs. 9 and 11 at least as associated Euler's equations. The functional, if it exists, should be composed of true and pseudoflow variables as explained by Flax[9] and Bessho and Nomura[10] (Fig. 3). Guided by them, we might have two more Euler's equations connected with the reverse flow. It is possible to

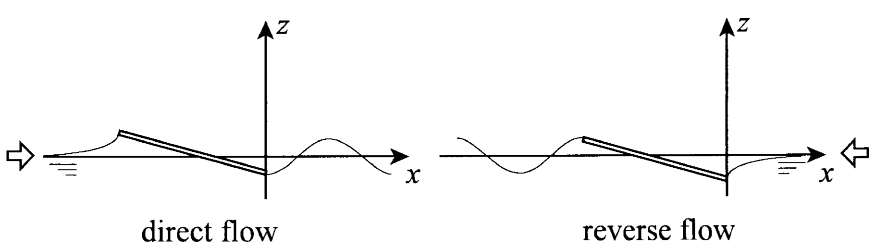

Fig. 3. Direct flow and reverse flow

construct Π by adding together the reverse flow variables $\bar{\mu}$ and $\bar{\Gamma}$ to normal ones μ and Γ. As a matter of course, l_w is also an unknown variable and accompanies the fifth Euler's equation but is redundant. As Matsumura and Mizutani[5] indicated, l_w is probably connected with Γ, which is determined by assuming Kutta's condition in a hydrodynamic sense. So the fifth equation is believed to correspond to Kutta's condition.

We state that the flow around a planing ship is realized if the following functional Π is stationary:

$$\begin{aligned}
&\Pi\left[\mu,\bar{\mu},\Gamma,\bar{\Gamma},l_w\right] \\
&\equiv \int_{-1}^{1}\int_{-l_w(y)}^{0}\left\{-\bar{\mu}(x,y)\frac{\partial H}{\partial x}(x,y) - \mu(x,y)\frac{\partial H}{\partial x}(x,y)\right\}dxdy \\
&+\int_{-1}^{1}\int_{-l_w(y)}^{0}\bar{\mu}(x,y)\left\{\fint_{-1}^{1} Q_x(x,y;0,\eta)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{x\xi}(x,y;\xi,\eta)\mu(\xi,\eta)d\xi d\eta\right\}dxdy \\
&+\int_{-1}^{1}\left\{-\bar{\Gamma}(y)H\left(-l_w(y),y\right)-\Gamma(y)H(0,y)\right\}dy \\
&+\int_{-1}^{1}\bar{\Gamma}(y)\left\{\fint_{-1}^{1} Q\left(-l_w(y),y;0,\eta\right)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l(\eta)}^{0} Q_\xi\left(-l_w(y),y;\xi,\eta\right)\mu(\xi,\eta)d\xi d\eta\right\}dy
\end{aligned} \tag{12}$$

under the restricted conditions

$$\mu(0,y)=\Gamma(y) \tag{13}$$

$$\bar{\mu}(0,y)=0 \tag{14}$$

The functional eq. 12 is defined on the unknown surface denoted by $l_w(y)$ which is common to normal and adjoint reverse flow. Equation 13 is Kutta's condition. Conversely, $\bar{\mu}(-l_w(y), y)$ is distinguished from $\bar{\Gamma}(y)$ since they are expected to be equal as a result of a natural condition of eq. 12, which means Kutta's condition in reverse flow. In reverse flow, we presume a vortex system generated inversely upstream at the transom stern by imposing eq. 14.

The first variation of eq. 12 becomes

$$\begin{aligned}
\delta\Pi &= \int_{-1}^{1}\int_{-l_w(y)}^{0}\delta\bar{\mu}(x,y) \\
&\quad\times\left\{-\frac{\partial H}{\partial x}(x,y)+\fint_{-1}^{1} Q_x(x,y;0,\eta)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{x\xi}(x,y;\xi,\eta)\mu(\xi,\eta)d\xi d\eta\right\}dxdy \\
&+\int_{-1}^{1}\delta\bar{\Gamma}(y) \\
&\quad\times\left\{-H\left(-l_w(y),y\right)+\fint_{-1}^{1} Q\left(-l_w(y),y;0,\eta\right)\Gamma(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_\xi\left(-l_w(y),y;\xi,\eta\right)\mu(\xi,\eta)d\xi d\eta\right\}dy \\
&+\int_{-1}^{1}\int_{-l_w(y)}^{0}\delta\mu(x,y) \\
&\quad\times\left\{-\frac{\partial H}{\partial x}(x,y)+\fint_{-1}^{1} Q_x\left(-l_w(\eta),y;x,\eta\right)\bar{\Gamma}(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{x\xi}(\xi,y;x,\eta)\bar{\mu}(\xi,\eta)d\xi d\eta\right\}dxdy \\
&+\int_{-1}^{1}\delta\Gamma(y) \\
&\quad\times\left\{-H(0,y)+\fint_{-1}^{1} Q\left(-l_w(\eta),y;0,\eta\right)\bar{\Gamma}(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_\xi(\xi,y;0,\eta)\bar{\mu}(\xi,\eta)d\xi d\eta\right\}dy \\
&+\int_{-1}^{1}\delta l(y)\Big[\mu\left(-l_w(y),y\right) \\
&\quad\times\left\{-\frac{\partial H}{\partial x}\left(-l_w(y),y\right) + \fint_{-1}^{1} Q_x\left(-l_w(\eta),y;-l_w(y),\eta\right)\bar{\Gamma}(\eta)d\eta - \fint_{-1}^{1}\int_{-l_w(\eta)}^{0} Q_{x\xi}\left(\xi,y;-l_w(y),\eta\right)\bar{\mu}(\xi,\eta)d\xi d\eta\right\} \\
&\quad-\left\{\bar{\Gamma}(y)-\bar{\mu}\left(-l_w(y),y\right)\right\} \\
&\quad\times\left\{-\frac{\partial H}{\partial x}\left(-l_w(y),y\right)+\fint_{-1}^{1} Q_x\left(-l_w(y),y;0,\eta\right)\Gamma(\eta)d\eta
\end{aligned} \tag{15}$$

All of the braced terms in eq. 15 become zero simultaneously when the functional is stationary. Therefore, the following conditions are satisfied:

1. the hull boundary condition (eq. 9) in normal flow, as seen in the first set of braces;
2. the spray condition (eq. 11) in normal flow, as seen in the second set of braces;
3. the hull boundary condition as eq. 9 in reverse flow, as seen in the third set of braces;
4. the spray condition as eq. 11 in reverse flow, as seen in the fourth set of braces.

Euler's equation, in the fifth term, can be separated into two parts, since neither braced expression becomes 0. The first part means that

$$\mu\left(-l_w(y),y\right)=0 \tag{16}$$

which is an implicitly assumed condition such that a vortex system in normal flow generates downstream at the spray root line. The second part means that

$$\bar{\mu}(-l_w(y), y) = \bar{\Gamma}(y) \tag{17}$$

i.e., Kutta's condition in reverse flow. Thus, all the required equations are satisfied in an antisymmetric manner with the help of reverse flow.

Determination of the wetted surface of a high aspect ratio planing plate

In order to show the effectiveness of the variational principle, we determine the form of the wetted surface of a high aspect ratio planing plate. We may assume that $K_0 << 1$, since the ship planes at high speed.

Expanding[2] the integrand of eq. 12 asymptotically to $0(K_0)$, we can express an approximate functional as follows:

$$\begin{aligned}
&\Pi_a[\mu, \bar{\mu}, \Gamma, \bar{\Gamma}, l] \\
&\equiv \int_{-1}^{1}\int_{-l_w(y)}^{0}\left\{-\bar{\mu}(x,y)\frac{\partial H}{\partial x}(x,y)\right. \\
&\qquad \left. -\mu(x,y)\frac{\partial H}{\partial x}(x,y)\right\}dxdy \\
&+\int_{-1}^{1}\int_{-l_w(y)}^{0}\bar{\mu}(x,y)\left\{-\frac{\Gamma(y)}{2\pi}\left(\frac{1}{x}+\frac{\pi}{2}K_0\right)\right. \\
&\qquad \left. +\frac{1}{2\pi}\oint_{-l_w(y)}^{0}\frac{\mu(\xi,y)}{(x-\xi)^2}d\xi-\frac{K_0}{2}\mu(x,y)\right\}dxdy \\
&+\int_{-1}^{1}\left\{-\bar{\Gamma}(y)H(-l_w(y),y)-\Gamma(y)H(0,y)\right\}dy \\
&+\int_{-1}^{1}\bar{\Gamma}(y)\left\{-\frac{\Gamma(y)}{2\pi}\left(\log(K_0 l_w(y))+\gamma\right)\right. \\
&\qquad \left. -\frac{1}{2\pi}\int_{-l_w(y)}^{0}\frac{\mu(\xi,y)}{-l_w(y)-\xi}d\xi+\frac{1}{4\pi}\oint_{-1}^{1}\frac{\Gamma(\eta)}{|y-\eta|}d\eta\right\}dy
\end{aligned} \tag{18}$$

where γ is Euler's constant.

We now think of a flat plate planing with trim τ, so that

$$H(x, y) = H_0 - \tau x \tag{19}$$

$$l_R(y) = l_R \equiv -\frac{H_0}{\tau} \tag{20}$$

A high aspect ratio approximation allows us to choose a simple trial function which is the same as the exact solution for a 2D flat wing:

$$\begin{aligned}
\mu(l_w(y)x^*, y) &\equiv \frac{2\Gamma(y)}{\pi}\mu^*(x^*) \\
&= \frac{2\Gamma(y)}{\pi}\int_{-1}^{x^*}\sqrt{\frac{-\xi}{1+\xi}}d\xi
\end{aligned} \tag{21}$$

$$\begin{aligned}
\bar{\mu}(l_w(y)x^*, y) &\equiv \frac{2\bar{\mu}(0,y)}{\pi}\bar{\mu}^*(x^*) \\
&= \frac{2\bar{\mu}(0,y)}{\pi}\int_{x^*}^{0}\sqrt{\frac{1+\xi}{-\xi}}d\xi
\end{aligned} \tag{22}$$

where

$$x \equiv l_w(y)x^* \tag{23}$$

Moreover, for the sake of simplicity, let us assume that the outline of the wetted surface is

$$l_w(y) = l_{w0}\sqrt{1-y^2} \tag{24}$$

and the circulation is

$$\Gamma(y) = \Gamma_0\sqrt{1-y^2} \tag{25}$$

$$\bar{\Gamma}(y) = \bar{\Gamma}_0\sqrt{1-y^2} \tag{26}$$

$$\bar{\mu}(0, y) = \mu_0\sqrt{1-y^2} \tag{27}$$

so that the restricted conditions in eqs. 13 and 14 are satisfied. Thus, Π_a depends on new unknown parameters μ_0, Γ_0, $\bar{\Gamma}_0$ and l_{w0}, which are the respective values at the center line.

It is easy to obtain the extremal of Π_a. The relationship between l_{w0} and Γ_0 is obtained by reclaiming $\partial\Pi_a/\partial\bar{\mu}_0 = 0$:

$$\begin{aligned}
\frac{\partial \Pi_a}{\partial \bar{\mu}} &= \int_{-1}^{1}\int_{-1}^{0} l_{w0}(1-y^2)\bar{\mu}^*(x^*) \\
&\quad \times\left\{\tau-\frac{\Gamma_0}{\pi l_{w0}}-K_0\Gamma_0\sqrt{1-y^2}\left(\frac{1}{4}+\frac{\mu^*(x^*)}{\pi}\right)\right\} \\
&\quad \times dx^*\,dy \\
&= 0
\end{aligned} \tag{28}$$

so that

$$\Gamma_0 \equiv \frac{\pi\tau l_0}{1+K_0 l_{w0} a} \tag{29}$$

where a is defined as

$$a = \frac{9\pi^2}{128} + \frac{3}{4}\int_{-1}^{0}\mu^*(x^*)\bar{\mu}^*(x^*)dx^* \tag{30}$$

Next, reclaiming $\partial\Pi_a/\partial\overline{\Gamma}_0 = 0$ subject to eq. 30, we get

$$\frac{\partial\Pi_a}{\partial\overline{\Gamma}} = \frac{\tau}{K_0}\left\{\frac{\pi}{2}\left(\frac{1}{Fn_r^2}\right) - \frac{4}{3}\left(\frac{1}{Fn^2}\right) + \frac{\frac{1}{Fn^2}}{1+\frac{1}{Fn^2}a} \times\left(-\frac{2}{3}\log\left(\frac{1}{Fn^2}\right) + \frac{8}{3}\log 2 - \frac{2}{3}\gamma - \frac{1}{3}\right)\right\} = 0 \quad (31)$$

where

$$\frac{1}{Fn_R^2} \equiv \frac{gl_R}{U_\infty^2} \qquad \frac{1}{Fn^2} \equiv \frac{gl_{w0}}{U_\infty^2} \quad (32)$$

As shown in Fig. 4, eq. 31 gives a simple relation between the wetted length in motion and that at a standstill, regardless of the trim angle. We now have three results.

1. Only one solution exists in the case where $1/Fn_R^2 > 0$.
2. Two solutions exist in the case where $-0.06 < 1/Fn_R^2 < 0$.
3. No solutions exist in the case where $1/Fn_R^2 < -0.06$.

The case where $Fn_R^2 < 0$, namely $l_R < 0$, is when the plate is set above a still water surface, even during planing.

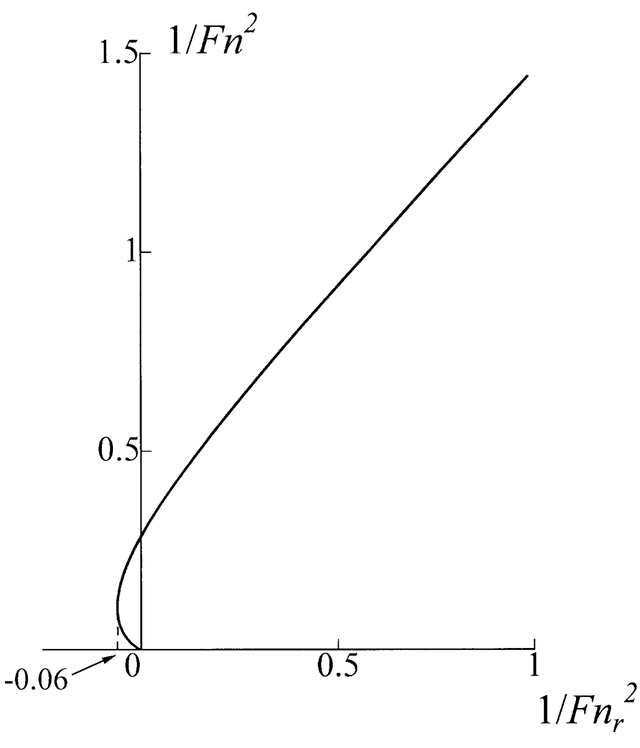

Fig. 4. Relation between the Froude number based on the still-water length and the Froude number based on the wetted length at the centerline in motion

Such a steady state can exist up to the maximum height shown in Fig. 4 by an initial disturbance of the water. In the second case, a larger wetted length may give a stable solution, as described by Matsumura and Mizutani[5] in gravity-free cases.

In Fig. 5, the experimental results[5] for a wetted surface and calculated values are shown for various conditions, denoted by l_R. The experimental results match well despite the simple calculation.

Conclusion

We have considered the unknown wetted surface problem of a planing ship under gravity and shown a variational principle associated with this problem.

This principle is characterized by the following points:

1. It is adjoined with the reverse flow.
2. The functional is expressed by a vortex line function and a downstream circulation distribution, which is distinguished from each other, in addition to the unknown wetted length distribution which defines the integration region.
3. Kutta's condition and the adjunct are imposed as constraints.

Through a simple example assuming high aspect ratio planing surfaces, we give the following summary.

4. The approximate functional is effective in estimating the wetted surface.
5. A simple relation, regardless of the trim angle, is found between the wetted length in motion and the one at a standstill.

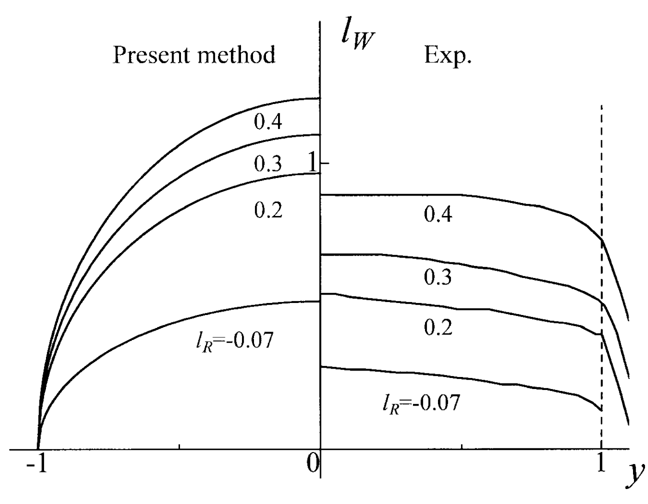

Fig. 5. Comparison of the calculated results of the wetted surface area of a planing plate with the experimental ones ($K_0 = 0.436$)

6. It even expresses the state of a planing plate set over a still water surface. The maximum height of the plane was obtained.

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